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On the Generalized Spencer Cohomology of Some Graded Lie Algebras I

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階別リー環の一般化されたスペンサーコホモロジーについて I

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Abstract. We consider maximal parabolic subalgebras $\mathfrak{p}$ of finite dimensional classical simple irreducible graded Lie algebras $\mathfrak{s}$ over $\mathbf{C}$, and give the description of the generalized Spencer cohomology space of the graded subalgebra $\mathfrak{p}$ of $\mathfrak{s}$.

Introduction. Let $\mathfrak{s}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{s}_p$ be a finite dimensional simple graded Lie algebra (SGLA) over $\mathbf{C}$ such that: (i) $\mathfrak{s}_{-1}\neq 0$; (ii) the negative part $\mathfrak{s}_-=\bigoplus_{p<0}\mathfrak{s}_p$ is generated by $\mathfrak{s}_{-1}$; (iii) $\mathfrak{s}$ is of classical type. Further let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{s}$ containing $\bigoplus_{p\leq 0}\mathfrak{s}_p$, and $\mathfrak{n}$ (resp. $\mathfrak{l}$) the nilradical (resp. a reductive Levi subalgebra) of $\mathfrak{p}$. Our main concern is to describe the cohomology space $H^*(\mathfrak{s}_-, \mathfrak{p})=\bigoplus_{s\in\mathbf{Z}} H^*(\mathfrak{s}_-, \mathfrak{p})_s$, which is called the generalized Spencer cohomology space of the graded Lie algebra $\mathfrak{p}$. It is known that the p-th cohomology space $H^p(\mathfrak{s}_-, \mathfrak{p})$ is isomorphic to $\bigoplus_{i=0}^{p} H^{p-i}(\mathfrak{l}_-, H^i(\mathfrak{n}, \mathfrak{p}))$ as an $\mathfrak{l}_0$-module, where $\mathfrak{l}_-=\mathfrak{l}\cap\mathfrak{s}_-$ and $\mathfrak{l}_0=\mathfrak{l}\cap\mathfrak{s}_0$. Therefore by Kostant's theorem ([Kos 61, Th. 5. 14]), our problem is reduced to that of describing the space $H^i(\mathfrak{n}, \mathfrak{p})$.

The purpose of this note is to describe the spaces $H^1(\mathfrak{n}, \mathfrak{p})$ and $H^2(\mathfrak{n}, \mathfrak{p})$ in case $\mathfrak{s}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{s}_p$ is irreducible and $\mathfrak{p}$ is a maximal parabolic subalgebra of $\mathfrak{s}$. The details of this note will be published elesewhere.

§ 1. Let $\mathfrak{s}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{s}_p$ be a finite dimensional SGLA over $\mathbf{C}$ satisfying the following conditions:

(P1) rank $\mathfrak{s}\geq 2$;

(P2) $\mathfrak{s}_{-1}\neq 0$ and the negative part $\mathfrak{n}:=\bigoplus_{p<0}\mathfrak{s}_p$ of $\mathfrak{s}$ is generated by $\mathfrak{s}_{-1}$;

(P3) $\mathfrak{s}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{s}_p$ is irreducible, that is, the $\mathfrak{s}_0$-module $\mathfrak{s}_{-1}$ is irreducible;

(P4) $\mathfrak{s}$ is of classical type.

Then the subalgebra $\mathfrak{p}=\bigoplus_{p\leq 0}\mathfrak{s}_p$ of $\mathfrak{s}$ is maximal parabolic and $\mathfrak{n}$ (resp. $\mathfrak{l}$) is the nilradical (resp. a reductive Levi subalgebra) of $\mathfrak{p}$, where $\mathfrak{l}=\mathfrak{s}_0$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{s}$ contained in $\mathfrak{s}_0$ and Δ the root system of $(\mathfrak{s}, \mathfrak{h})$. For a $\mathfrak{h}$-stable subspace $\mathfrak{a}$ of $\mathfrak{s}$, we denote by $\Delta(\mathfrak{a})$ the set of all the roots in which the corresponding root vectors are contained in $\mathfrak{a}$. Then there exist a simple root system $\Pi=\{\alpha_1, \ldots, \alpha_l\}$ (l=rank $\mathfrak{s}$) of Δ and an l-tuple $\mathbf{s}=(s_1, \ldots, s_l)$ of non-negative integers such that $\Delta(\mathfrak{s}_p)=\{\alpha$

$\in \Delta$: $\ell_s(\alpha)=p\}$, where $\ell_s(\Sigma k_i\alpha_i)=\Sigma k_i s_i$. If $\mathfrak{s}$ is a simple Lie algebra of type X_l, then the SGLA $\mathfrak{s}=\bigoplus_{p\in Z}\mathfrak{s}_p$ is said to be of type (X_l, s). In what follows, we label the Dynkin diagram as in [Bou 68] .

Let $V=\bigoplus_{p\in Z} V_p$ be a graded $\mathfrak{n}$-module (i. e., V is a vector space with a gradation $(V_p)_{p\in Z}$ such that $\mathfrak{s}_p\cdot V_q\subset V_{p+q}$.), $C^p(\mathfrak{n}, V)=\mathrm{Hom}(\wedge^p\mathfrak{n}, V)$ the p-chains of the $\mathfrak{n}$-module and $H^p(\mathfrak{s}, V)$ the space of the p-th cohomology with respect to the coboundary operator ∂. The gradations of $\mathfrak{n}$ and V define a gradation in the space of cochains:

$$C^p(\mathfrak{n}, V)=\bigoplus_{s\in Z} C^p(\mathfrak{n}, V)_s,$$

where $C^p(\mathfrak{n}, V)_s=\{\omega\in C^p(\mathfrak{n}, V): \omega(\mathfrak{s}_{i_1}\wedge\cdots\wedge\mathfrak{s}_{i_p})\subset V_{i_1+\cdots+i_p+s}, i_1,\ldots,i_p<0\}$. This gradation is compatible with the coboundary operator ∂, and we obtain a gradation of the cohomology space:

$$H^p(\mathfrak{n}, V)=\bigoplus_{s\in Z} H^p(\mathfrak{n}, V)_s.$$

Furthermore if V has a $\mathfrak{p}$-module structure, then a $\mathfrak{p}$-module structure on $C^p(\mathfrak{n}, V)$ can be introduced in the natural way. Since the coboundary operator ∂ is a $\mathfrak{p}$-module homomorphism, the space $H^p(\mathfrak{n}, V)$ also acquires a $\mathfrak{p}$-module structure. In particular, each $H^p(\mathfrak{n}, V)_s$ is an $\mathfrak{l}$-submodule of $H^p(\mathfrak{n}, V)$. In case V is an irreducible $\mathfrak{s}$-module, according to Kostant's theorem, we can decompose $H^p(\mathfrak{n}, V)$ into a direct sum of irreducible $\mathfrak{l}$-modules. In particular, we can obtain the description of the cohomology space $H^p(\mathfrak{n}, \mathfrak{s})$. In § 2, we will give the description of the space $H^i(\mathfrak{n}, \mathfrak{p})$ $(i=1, 2)$ by using the spaces $H^i(\mathfrak{n}, \mathfrak{s})$ and well-known $\mathfrak{l}_0$-modules. In what follows, for $\mathfrak{l}$-modules M and N, $M\cong N$ will mean that M and N are isomorphic to each other.

§ **2.** Let $\mathfrak{s}$, $\mathfrak{p}$ and $\mathfrak{n}$ be as in § 1. Assume that $\mathfrak{s}=\bigoplus_{p\in Z}\mathfrak{s}_p$ is of depth ν, that is, $\mathfrak{s}_{-\nu}\neq 0$ and $\mathfrak{s}_p=0$ for $p<-\nu$. Let W be the Weyl group of $\mathfrak{s}$ and Δ_+ the set of positive roots. Now we introduce the following notations:

$$T_w=w\Delta_-\cap\Delta_+\ (w\in W),$$
$$W^j(\mathrm{s})=\{w\in W: \#(T_w)=j,\ T_w\subset\Delta(\mathfrak{s}_+)\},$$
$$W^j(\mathfrak{p})=\{w\in W^j(\mathrm{s}): -w\theta\in\Delta(\mathfrak{p})\},$$
$$\varepsilon_i=(\delta_{1i},\ldots,\delta_{li}),$$

where $\Delta_-=\Delta\backslash\Delta_+$ and θ is the highest root of $\mathfrak{s}$. Let

$$0\longrightarrow\mathfrak{p}\xrightarrow{\phi}\mathfrak{s}\xrightarrow{\psi}\mathfrak{s}/\mathfrak{p}\longrightarrow 0$$

be a short exact sequence, where ϕ is an embedding and ψ is the natural homomorphism. This sequence generates a long exact sequence,

$$\cdots\longrightarrow H^{j-1}(\mathfrak{n}, \mathfrak{s}/\mathfrak{p})\xrightarrow{\delta^*_{j-1}} H^j(\mathfrak{n}, \mathfrak{p})\xrightarrow{\phi^*_j} H^j(\mathfrak{n}, \mathfrak{s})\xrightarrow{\psi^*_j} H^j(\mathfrak{n}, \mathfrak{s}/\mathfrak{p})\longrightarrow\cdots.$$

By Kostant's theorem, if $W^j(\mathfrak{p})=W^j(\mathrm{s})$ (resp. $W^j(\mathfrak{p})=\emptyset$), then $\mathrm{Im}\,\phi^*_j=H^j(\mathfrak{n}, \mathfrak{s})$ (resp. $\mathrm{Im}\,\phi^*_j=0$). Since $W^1(\mathrm{s})=W^1(\mathfrak{p})$, we get the following lemma:

Lemma 1. (1) $H^1(\mathfrak{n}, \mathfrak{p})_1\cong\mathfrak{s}_1\oplus H^1(\mathfrak{n}, \mathfrak{s})_1$;

(2) $H^1(\mathfrak{n}, \mathfrak{p})_s\cong H^1(\mathfrak{n}, \mathfrak{s})_s$ *for all* $s\neq 1$;

(3) $H^2(\mathfrak{n}, \mathfrak{p})_s \cong (\mathrm{Im}\phi_2^*)_s \oplus H^1(\mathfrak{n}, \mathfrak{s}/\mathfrak{p})_s$, *where* $(\mathrm{Im}\ \phi_2^*)_s = \mathrm{Im}\ \phi_2^* \cap H^2(\mathfrak{n}, \mathfrak{s})_s$.

Next we investigate the space $H^2(\mathfrak{n}, \mathfrak{p})$. First note that $\mathfrak{s}/\mathfrak{p}$ is isomorphic to $\mathfrak{n}^*$ as a $\mathfrak{p}$-module and $H^1(\mathfrak{n}, \mathfrak{s}/\mathfrak{p})_s = 0$ for $s \leq 1$. Further we see that $W^2(\mathfrak{s}) = W^2(\mathfrak{p})$ if one of the following conditions holds:

(1) rank $\mathfrak{s} \geq 3$;

(2) $(X_l, \mathbf{s}) \neq (A_2, \varepsilon_1), (C_2, \varepsilon_2)$.

On the other hand, if $(X_l, \mathbf{s})$ is either (A_2, ε_1) or (C_2, ε_2), then $W^2(\mathfrak{p}) = \emptyset$ and $H^2(\mathfrak{n}, \mathfrak{s})_s = 0$ for $s \neq 3$. Hence, by Lemma 1 (3), we have the following lemma:

Lemma 2. (1) $H^2(\mathfrak{n}, \mathfrak{p})_s \cong H^2(\mathfrak{n}, \mathfrak{s})_s$ *for* $s \leq 1$.

(2) *If* $\nu = 1$, *then* $H^2(\mathfrak{n}, \mathfrak{p})_2 \cong H^2(\mathfrak{n}, \mathfrak{s})_2 \oplus H^1(\mathfrak{n}, \mathfrak{s}/\mathfrak{p})_2$ *and* $H^2(\mathfrak{n}, \mathfrak{p})_s = 0$ *for* $s \geq 3$.

(3) *If* $\nu \geq 2$, *then* $H^2(\mathfrak{n}, \mathfrak{p})_s \cong H^2(\mathfrak{n}, \mathfrak{s})_s \oplus H^1(\mathfrak{n}, \mathfrak{s}/\mathfrak{p})_s$ *for* $s \geq 2$.

We consider the spaces $H^1(\mathfrak{n}, \mathfrak{s}/\mathfrak{p})_s$. According to Theorem 4.1 in [LL 74, Th. 4.1], we obtain the following exact sequence

$$0 \longrightarrow H^0(\mathfrak{n}, S^2(\mathfrak{n}^*)) \longrightarrow H^1(\mathfrak{n}, \mathfrak{n}^*) \longrightarrow H^2(\mathfrak{n}, \mathbf{C}) \longrightarrow 0.$$

Hence

$$\begin{aligned} \mathrm{ch}_{\mathfrak{l}}\ (H^1(\mathfrak{n}, \mathfrak{s}/\mathfrak{p})) &= \mathrm{ch}_{\mathfrak{l}}\ (H^1(\mathfrak{n}, \mathfrak{n}^*)) \\ &= \mathrm{ch}_{\mathfrak{l}}\ (H^0(\mathfrak{n}, S^2\ (\mathfrak{n}^*))) + \mathrm{ch}_{\mathfrak{l}}\ (H^2\ (\mathfrak{n}, \mathbf{C})). \end{aligned}$$

Also we can easily see that:

(i) $H^0(\mathfrak{n}, S^2(\mathfrak{n}^*))_2 = S^2(\mathfrak{s}_{-1}^*)$;

(ii) $\mathrm{ch}_{\mathfrak{l}}(H^2(\mathfrak{n}, \mathbf{C})_2) = \mathrm{ch}_{\mathfrak{l}}(\wedge^2\ \mathfrak{s}_{-1}^*) - \mathrm{ch}_{\mathfrak{l}}(\mathfrak{s}_{-2}^*)$.

Now we assume $\nu \geq 2$. Let

$$0 \longrightarrow \mathfrak{p}/\mathfrak{n} \longrightarrow \mathfrak{s}/\mathfrak{n} \longrightarrow \mathfrak{s}/\mathfrak{p} \longrightarrow 0$$

be a short exact sequence of $(\mathfrak{z}\ (\mathfrak{l}) + \mathfrak{n})$-modules, where $\mathfrak{z}\ (\mathfrak{l})$ is the center of $\mathfrak{l}$. This sequence generates a long exact sequence

$$\cdots \longrightarrow H^1(\mathfrak{n}, \mathfrak{p}/\mathfrak{n})_s \longrightarrow H^1(\mathfrak{n}, \mathfrak{s}/\mathfrak{n})_s \longrightarrow H^1(\mathfrak{n}, \mathfrak{s}/\mathfrak{p})_s \longrightarrow H^2(\mathfrak{n}, \mathfrak{p}/\mathfrak{n})_s \longrightarrow \cdots.$$

Since the $(\mathfrak{z}(\mathfrak{l}) + \mathfrak{n})$-module $\mathfrak{p}/\mathfrak{n}$ is trivial, $H^2\ (\mathfrak{n}, \mathfrak{p}/\mathfrak{n})_s$ is isomorphic to a direct sum of dim $\mathfrak{p}/\mathfrak{n}$-copies of $H^2(\mathfrak{n}, \mathbf{C})_s$ as an $\mathfrak{n}$-module. Also $H^1(\mathfrak{n}, \mathfrak{s}/\mathfrak{n})_s = 0$ for $s \geq 3$ ([Kha 90, p.111]). Therefore, if $H^2(\mathfrak{n}, \mathbf{C})_s = 0$ $(s \geq 3)$, then

$$\mathrm{ch}_{\mathfrak{l}}(H^2(\mathfrak{n}, \mathfrak{p})_s) = \mathrm{ch}_{\mathfrak{l}}(H^2(\mathfrak{n}, \mathfrak{s})_s)\ (s \geq 3).$$

By Kostant's theorem, in order that $H^2(\mathfrak{n}, \mathbf{C})_s \neq 0$ for some $s \geq 3$, it is necessary that the pair $(X_l, \mathbf{s})$ be either (C_l, ε_{l-1}) or (B_l, ε_l). In this case, we have

Lemma 3. *Assume that* $\nu \geq 2$. *Then*:

(1) *Let* $(X_l, \mathbf{s}) = (C_l, \varepsilon_{l-1})$. *Then* $H^2(\mathfrak{n}, \mathbf{C})_s = 0$ *for* $s \geq 4$ *and* $H^0(\mathfrak{n}, S^2\ (\mathfrak{n}^*))_3 = 0$.

(2) *Let* $(X_l, \mathbf{s}) = (B_l, \varepsilon_l)$. *Then* $H^2(\mathfrak{n}, \mathbf{C})_s = 0$ *for* $s \geq 4$ *and* $H^0(\mathfrak{n}, S^2(\mathfrak{n}^*))_3 \cong \wedge^3\ \mathfrak{s}_{-1}^*$.

By summarizing the results of Lemmas 1-3, we get the following main theorem.

Theorem 4. *Let* $\mathfrak{s}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{s}_p$ *be a complex SGLA of type* (X_l, s). *Assume that* $\mathfrak{s}=\bigoplus_{p\in\mathbf{Z}}\mathfrak{s}_p$ *is of depth* ν *and the conditions (P1)*$-$*(P4) hold. We set* $\mathfrak{p}=\bigoplus_{p\leq 0}\mathfrak{s}_p$ *and* $\mathfrak{n}=\bigoplus_{p<0}\mathfrak{s}_p$. *Then:*

(1) $H^1(\mathfrak{n}, \mathfrak{p})_s \cong H^1(\mathfrak{n}, \mathfrak{s})_s$ *for all* $s\neq 1$;

(2) $H^1(\mathfrak{n}, \mathfrak{p})_1 \cong H^1(\mathfrak{n}, \mathfrak{s})_1 \oplus \mathfrak{s}_1$;

(3) $H^2(\mathfrak{n}, \mathfrak{p})_s \cong H^2(\mathfrak{n}, \mathfrak{s})_s$ *for all* $s\leq 1$;

(4) $H^2(\mathfrak{n}, \mathfrak{p})_2 \cong H^2(\mathfrak{n}, \mathfrak{s})_2 \oplus \bigotimes_0^2 \mathfrak{s}_{-1}^*$, *where* $\bigotimes_0^2 \mathfrak{s}_{-1}^*$ *is the* $\mathfrak{l}$*-submodule of* $\bigotimes^2 \mathfrak{s}_{-1}^*$ *such that* $\mathrm{ch}_{\mathfrak{l}}(\bigotimes_0^2 \mathfrak{s}_{-1}^*)=\mathrm{ch}_{\mathfrak{l}}(\bigotimes^2 \mathfrak{s}_{-1}^*)-\mathrm{ch}_{\mathfrak{l}}(\mathfrak{s}_{-2}^*)$;

(5) *In case* $\nu=1$, $H^2(\mathfrak{n}, \mathfrak{p})_s=0$ *for* $s\geq 3$;

(6) *In case* $\nu\geq 2$ *and except for the case when* $(X_l, \mathrm{s})=(C_l, \varepsilon_{l-1})$, (B_l, ε_l), *we get* $H^2(\mathfrak{n}, \mathfrak{p})_s \cong H^2(\mathfrak{n}, \mathfrak{s})_s$ *for* $s\geq 3$;

(7) *Let* $(X_l, \mathrm{s})=(C_l, \varepsilon_{l-1})$. *Then* $H^2(\mathfrak{n}, \mathfrak{p})_s \cong H^2(\mathfrak{n}, \mathfrak{s})_s$ *for* $s\geq 4$, *and* $H^2(\mathfrak{n}, \mathfrak{p})_3 \cong H^2(\mathfrak{n}, \mathfrak{s})_3 \oplus H^2(\mathfrak{n}, \mathbf{C})_3$.

(8) *Let* $(X_l, \mathrm{s})=(B_l, \varepsilon_l)$. *Then* $H^2(\mathfrak{n}, \mathfrak{p})_s \cong H^2(\mathfrak{n}, \mathfrak{s})_s$ *for* $s\geq 4$, *and* $H^2(\mathfrak{n}, \mathfrak{p})_3 \cong H^2(\mathfrak{n}, \mathfrak{s})_3 \oplus (\wedge^2 \mathfrak{s}_{-1}^* \otimes \mathfrak{s}_{-1}^*)$.

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