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(4) H. Hombu Die Krümmungstheorie im Finslerschen Räume, J. F. S. Hok 5 1936

(5) M. Haimovici Formules fondamentals dans la théorie des Hypersurfaces d'un espace de Finsler, C. R. 198 1934

(6) J. M. Wegener Hyperflächeh in Finslerschen Räumen als Transversalflächen einer Schar von Extremalen, Monatsh 44 1936

(7) E. T. Davies Subspace of a Finsler Space, Proc. London. Math. So 49 1945

(8) M. Haimovici Variétés totalément et variétés totalement géodésiques dans les espace de Finsler, Annales. Scientifiques de L'université de Jassy 25 1939

(9) J. A. Schouten Einführung in die neueren Methoden der Differential Geometrie, Band I, II, 1935

On the Theorem of Schur in the Special Cartan Space.

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矢野晋平 : 特殊 Cartan 空間に於ける Schur の定理について

The purpose of this paper is to explain that the theorem of *Schur* in the Riemannian geometry [1] holds good again for the special *Cartan* space in which the fundamental conditions are given, that is

$$(1) \qquad \Gamma^{*i}_{j\,k}\|^{l} = 0, \qquad R_{iojk} = 0.$$

L. BERWALD showed in his paper [2], that the necessary and sufficient conditions under which the Cartan space is the same as *Minkowski* space are that $\Gamma^{*i}_{j\,k}\|^{l}=0$ and $R_{ijkl}=0$. Accordingly, our space may be said a more general space than the Minkowski space.

§ 1. *The curvature tensor of the space.* From the fundamental conditions (1) and the identities explained by L. BERWALD (Acta, (15. 1), (17. 1), (12. 12), (12. 7), (12. 10), (18. 8) and (18. 10)) we have

$$(2) \qquad \bar{R}_{i\ hk}^{\ j} = 2\Gamma^{*j}_{i\ [h,k]} + 2\Gamma^{*a}_{i\ [h}\Gamma^{*\ j}_{|a|\ k]},$$

$$(3) \qquad R_{i\ hk}^{\ j} = \bar{R}_{i\ hk}^{\ j}, \qquad P_{ijh}^{\ \ \ k} = 0,$$

$$(4) \qquad A_{ij}^{\ \ k}|_{h} = 0, \qquad R_{ij(hk|l)} = 0,$$

$$(5) \qquad R_{ijhk}|^{l} - 2R_{ija(h}A_{k)}^{\ \ al} = 0,$$

where $R_{ijhk}|^{l} = R_{ijhk}\|^{l} - R_{ajhk}A_{i}^{\ al} - R_{iahk}A_{j}^{\ al} - R_{ijak}A_{h}^{\ al} - R_{ijha}A_{k}^{\ al}$,

$\partial\Gamma^{*j}_{i\ h}/\partial x^{k} = \Gamma^{*j}_{i\ k,h}$. Throughout this paper we shall use this notation : The first relation of (4) is rewritten in the following equation

$$(6) \qquad A_{ij\ ,h}^{\ \ k} + A_{ij}^{\ \ k}\|^{a}\Gamma^{*}_{aoh} - A_{aj}^{\ \ k}\Gamma^{*a}_{i\ h} - A_{ia}^{\ \ k}\Gamma^{*a}_{j\ h} + A_{ij}^{\ \ a}\Gamma^{*k}_{a\ h} = 0.$$

From (2), (3) and the second condition of (1), considering the relation (Acta, (8. 6)) $l_{j,k} = l_j\,\Gamma^{*}_{aok}(l^{a} - A^{a})$, we have

$$(7)\qquad \Gamma^{*}_{io[k,\,l]}=\Gamma^{*a}_{i\ [k}\{l_{|a}(l^{\beta}-A^{\beta})\Gamma^{*}_{\beta o|l]}-\Gamma^{*}_{|ao|l]}\}.$$

Considering the known relation (Acta, (5.14)) $(L/\sqrt{g})\|^{i}=L/\sqrt{g}\,(l^{i}-A^{i})$, for any kind of tensor $T^{a\cdots\beta}_{\gamma\cdots\delta}$, we have a general relation $T^{a\cdots\beta}_{\gamma\cdots\delta}\|^{[jk]}=T^{a\cdots\beta}_{\gamma\cdots\delta}\|^{[j}(l^{k]}-A^{k]})$. If we put the metric tensor g_{ij} in it, we have

$$(8)\qquad A_{ij}{}^{[a}\|^{\beta]}=A_{ij}{}^{[a}l^{\beta]}-A^{\beta]}).$$

If the obtained equation (Acta (6. 8)) $\Gamma^{*j}_{i\ k}=\Gamma^{\ j}_{i\ k}+A_{i}^{\ ja}\ \Gamma_{aok}$, be multiplied by l_j and $(\delta^{i}_{h}-l_{h}A^{i})$, and i and j be contracted, considering the relation $(\delta^{a}_{i}+l_{i}A^{a})(\delta^{i}_{h}-l_{h}A^{i})=\delta^{a}_{h}$, we may solve the Γ_{hok} conversely, that is

$$(9)\qquad \Gamma_{hok}=\Gamma^{*}_{iok}\ (\delta^{i}_{h}-l_{h}A^{i}).$$

If we insert Γ_{hok} into the known equations (Acta (7. 1), (7.3)) such as

$$g_{ij,k}=2\Gamma^{*}_{(ij)k}-2A_{ij}^{\ a}\Gamma_{aok}\quad\text{and}\quad \Gamma^{*}_{ijk}=\gamma_{ijk}+A_{ij}^{\ a}\ \Gamma_{aok}-A_{ik}^{\ a}\ \Gamma_{aoj}+A_{jk}^{\ a}\Gamma_{aoi}$$

$$\text{where}\qquad \gamma_{ijk}=\frac{1}{2}(g_{ij,k}+g_{jk,i}-g_{ik,j}),$$

we obtain again the formal equations

$$(10)\qquad g_{ij,k}=2\Gamma^{*}_{(ij)k}-2A_{ij}^{\ a}\Gamma^{*}_{aok}.$$

$$(11)\qquad \Gamma^{*}_{ijk}=\gamma_{ijk}+A_{ij}^{\ a}\Gamma^{*}_{aok}-A_{ik}^{\ a}\Gamma^{*}_{aoj}+A_{jk}^{\ a}\Gamma^{*}_{aoi}.$$

If we put the equations (10), (11) in the relation (2), in the following form, $A_{ij}^{\ a}\Gamma^{*}_{aok}-A_{ik}^{\ a}\Gamma^{*}_{aoj}+A_{jk}^{\ a}\Gamma^{*}_{aoi}=T_{ijk}$, we have

$$(12)\qquad R_{ijhk}=2(\gamma_{ij\,[h,k]}+T_{ij\,[h,k]}+2A_{aj}^{\ \ \beta}\ \Gamma^{*}_{\beta o[k}\Gamma^{*a}_{\ h]i}-\Gamma^{*a}_{i\,[h}\ \Gamma^{*}_{k]aj}).$$

Considering the relation $\gamma_{ij[h,k]}=\gamma_{hk[i,j]}$ we have the following equations

$$R_{ijhk}-R_{hkij}=2(A_{ij,[k}^{\ \ a}\Gamma^{*}_{|ao|h]}+A_{ij}^{\ a}\Gamma^{*}_{ao\,[h,k]}-A_{i[h,k]}^{\ \ a}\ \Gamma^{*}_{aoj}-A_{i[h}^{\ \ a}\ \Gamma^{*}_{|aoj|,k]}$$

$$+A_{j[h,k]}^{\ \ a}\ \Gamma^{*}_{aoi}+A_{j[h}^{\ \ a}\Gamma^{*}_{|aoi|,k]}+2A_{aj}^{\ \ \beta}\ \Gamma^{*}_{\beta o[k}\Gamma^{*a}_{|i|h]})\quad cycle(ih)(jk).$$

If we put the relations (6), (7) and (8) in the above equations and after lengthy calculations, we find that the right side is zero. Thus we have the

Theorem. In the special Cartan space given the fundamental conditions (1), we have the same relation as in the Riemannian space

$$(13)\qquad R_{ijhk}=R_{hkij}.$$

§ 2 *The theorem of Schur.* As in the Riemannian geometry we define the curvature K at a point and an element of hypersurface u_i for the orientation determined by two vectors $V_1{}^{i}$ and $V_2{}^{k}$ as follows

$$K=R_{ijkl}\ V_1{}^{i}V_2{}^{j}V_1{}^{k}V_2{}^{l}/(g_{ik}g_{jl}-g_{il}g_{jk})V_1{}^{i}V_2{}^{j}V_1{}^{k}V_2{}^{l}.$$

From (13) it follows that a necessary and sufficient condition under which the curvature at every point and element of hypersurface of our space be independent of the orientation is that

$$(14)\qquad R_{hjkl} = -K(g_{hk}g_{jl} - g_{hl}g_{jk}).$$

If we differentiate covariantly the equation (14) with i and put in the second relation of (4), and multiply by g^{hk} and g^{jl}, and contract the same indexes, we have the relation

$$(15)\qquad K|_i = K,_i + k\|^a \Gamma^*_{aoi} = 0.$$

Similarly, differentiate with u_i, and put in (5), we have

$$K|^l\,(g_{il}g_{jk} + g_{ik}g_{jh}) + 2R_{ija(h}\,A_{k)}{}^{al} = 0.$$

If the above equation be multiplied by l^i and i be contracted, from the second condition of (1) we have

$$(16)\qquad K|^l = (L/\sqrt{g})\,\partial K/\partial u_l = 0.$$

From (15) and (16) we have finally $K = const.$ Thus we have the

Theorem. The theorem of Schur in the Riemannian geometry holds good again in the special Cartan space in which the fundamental conditions (1) are given.

(October, *1951*)

REFERENCES

〔1〕 EISENHART, L. P. : Riemannian geometry, Princeton University press (*1926*)

〔2〕 BERWALD, L. : Über die *n*-dimensionalen Cartansche Räume und eine Normalform der zweiten Variation eines (*n-1*)-fachen Oberflächenintegrals (§ *22*), Acta mathemetica *71* (*1939*).

On the Molecular Weight of Liquid Polymers.

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清 水　清：液体重合体の分子量に就て

§ 1 Sound velocity in liquid became useful for the investigation of molecular structure after the discovery by M. R. Rao[1] of the empirical constant (called the molar sound velocity)

$$R = M\frac{v^{\frac{1}{3}}}{d} \qquad (1)$$

where M, v, and d are the molecular weight, sound velocity, and density.

Further, R.T. Lagemann and W.S. Dunbar[2] showed that, within a homologous series of liquid polymers, linear relationships exist between any two molar constants. An example of such a relationship is the one involving molar refraction N and molar sound velocity R

$$R = AN + B \qquad (2)$$

where A and B are, respectively, the slope and intercept.

Recently, A. Weissler, S. W. Fitzgerald,