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on a special Kawaguchi space

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北村五郎；特殊河口空間について

The fundamental theory of n-dimensional spaces such that the arc length of a curve $x^i = x^i(t)$ is given by the integral $s = \int \left\{ A^i(x, x') x''^i + B(x, x') \right\}^{\frac{1}{p}} dt$ has been established by Prof. Kawaguchi[1] and thereafter studied by many others.

In the present paper we study a generalization of an absolute derivative with $p \neq 1, 3$ when the components of a vector v^i are homogeneous of degree h with regard to the x' (§ I) and lead a necessary and sufficient condition that the special Kawaguchi space of even dimensions with $|H_{ij}| \neq 0$ be locally flat by another method from Watanabe's proof[2] (§ 2).

The notations are the same as those of Kawaguchi's paper.

§ I. A generalization of an absolute derivative.

The definitions of the absolute derivative

$$\delta v^i = dv^i + \Gamma^i_{(j)(k)} v^j dx^k$$

have no geometrical meaning if v^i are homogeneous of degree h with regard to the x. For we obtain

$$(\delta v^i)_{\bar t} = \sigma^h \delta v^i + h v^i \sigma^{h-1} d\sigma \qquad \left(\sigma = \frac{dt}{d\bar t} \right)$$

by transformation of the parameter t : $t = t(\bar t)$.

When we put

$$\delta^* v^i = \delta v^i - h v^i f(t),$$

we get

$$(\delta^* v^i)_{\bar t} = \sigma^h \{ \delta v^i - h v^i (f)_{\bar t} \} + h v^i \sigma^{h-1} d\sigma$$

by transformation of t.

Therefore f must has the following form by transformation t

$$(f)_{\bar t} = f + \frac{d\sigma}{\sigma} .$$

On the other hand

$$(\delta A_i x^{(2)i})_{\bar t} = \sigma^p \delta A_i x^{(2)i} + (p-2) F \sigma^{p-1} d\sigma - \sigma^{p-2} \frac{d\sigma}{d\bar t} A_i \delta x'^i,$$

$$(T_i \delta x'^i)_{\bar t} = \sigma^p T_i \delta x'^i + pF \sigma^{p-1} d\sigma - (2p-3) \sigma^{p-2} \frac{d\sigma}{d\bar t} A_i \delta x'^i .$$

Accodingly we get

$$\left(\frac{\delta A_i x^{(2)i}}{pF} \right)_{\bar t} = \frac{\delta A_i x^{(2)i}}{pF} + \frac{p-2}{p} \frac{d\sigma}{\sigma} - \frac{1}{pF} \sigma^{-2} \frac{d\sigma}{d\bar t} A_i \delta x'^i ,$$

$$\left(\frac{T_l\, \delta x'^l}{(2p-3)\, pF}\right)_{\bar{t}} = \frac{T_l\, \delta x'^l}{(2p-3)\, pF} + \frac{1}{2p-3}\, \frac{d\sigma}{\sigma} - \frac{1}{pF}\, \sigma^{-2}\, \frac{d\sigma}{d\bar{t}}\, A_l\, \delta\, x'^l ,$$

$$\therefore \quad \left(\frac{(2p-3)\, \delta\, A_l\, x^{(2)l} - T_l\, \delta\, x'^l}{(2p-3)\, pF}\right)_{\bar{t}} = \frac{(2p-3)\, \delta\, A_l\, x^{(2)l} - T_l\, \delta\, x'^l}{(2p-3)\, pF} + \left(\frac{p-2}{p} - \frac{1}{2p-3}\right) \frac{d\,\sigma}{\sigma} .$$

Consequently we can put

$$f = \frac{(2p-3)\, \delta\, A_l\, x^{(2)l} - T_l\, \delta\, x'^l}{2\,(p^2+1)\, F} .$$

Using the relations : $x^{(2)k} = -T_l\, G^{lk}$, $\quad G_{lk} = 2\, A_{l(k)} - A_{k(l)}$,
we put

$$\lambda_l = \frac{2p-3}{2(p^2+1)\, F}\, x^{(2)k}\, \nabla_l\, A_k ,$$

$$\mu_l = \frac{1}{(p^2+1) F}\, x^{(2)k} \left\{ (p-2)\, A_{k(l)} + A_{l(k)} \right\} .$$

Then we obtain as the required absolute differential

$$\delta^*\, v^l = \delta\, v^l - h v^l\, (\lambda_j\, dx^j + \mu_j\, \delta\, x'^j) ,$$

accompanied by the covariant derivatives

$$\nabla^*_j\, v^l = \nabla_j\, v^l - \frac{2p-3}{2(p^2+1)\, F}\, h v^l\, x^{(2)k}\, \nabla_j\, A_k ,$$

$$\nabla^{*\prime}_j\, v^l = \nabla'_j\, v^l - \frac{1}{(p^2+1)\, F}\, h v^l\, x^{(2)k} \left\{ (p-2)\, A_{k(j)} + A_{j(k)} \right\} .$$

$F^{-\frac{h}{p}}\, \delta^* v^l$ are indepent of a change of the parameter t.

§ 2. Another proof of the theorem.

Theorem : A necessary and sufficient condition that the special Kawaguchi space of even dimensions with $|H_{ij}| \neq 0$ be locally flat is that the following relations are satisfied :

$$A_{l(j)(k)} = 0, \quad R_{jkl}^{\cdots i} = 0, \quad B_{jkl}^{\cdots i} = 0, \quad \delta\, A_{l(j)} = 0 .$$

Proof. Necessity.

If the space be locally flat, the relations $A_{a(b)} = \text{const.}$, $B(y, y') = 0$ are satisfied in the suitable chosen y coordinate system.

Hereafter we use the indexes $i, j, k, \ldots ;\ a, b, c, \ldots$ in regard to x and y coordinate system respectively.

We can lead easily the relation $A_{l(j)(k)} = 0$ from $A_{a(b)} = \text{const.}$.

Differentiating $A_{a(b)}\, y'^b = (p-2)\, A_a$ with respect to y'^c, we get $p = 3$.

Consequently we have the following relations :

$$A_{a(b)}\, y'^b = A_a , \qquad A_{b(a)}\, y'^b = -A_a , \qquad \therefore\ A_{b(a)} = -A_{a(b)} .$$

Besides

$$H_{ab} = 2\, A_{a(b)} , \qquad G_{ab} = 3\, A_{a(b)} = \text{const.},$$

$$2\, \Gamma^c = (2\, A_{ab}\, y'^b - B_{(a)})\, G^{ac}$$

$$= (A_a\, G^{ab})_b\, y'^b = 2 \left(\frac{y'^c}{2p-3}\right)_b y'^b$$

$$= 0 .$$

Hence we have $R_{jkl}^{\cdots i}=0$, $B_{jkl}^{\cdots i}=0$ from $\Gamma^c=0$ and $\delta A_{a(b)}=0$ from $A_{a(b)}=$ const..

Sufficiency.

From $B_{jkl}^{\cdots i}=0$, then $\Gamma^i_{(j)(k)}$ are functions only of the x' s. If we consider now the simultaneous partial differential equations

$$\begin{cases} \dfrac{\partial y^a}{\partial x^i}=P_i^a \\ \dfrac{\partial P_i^a}{\partial x^j}=P_m^a\,\Gamma^m_{(i)(j)}\,, \end{cases}$$

these equations satisfy the integrability conditions from $R_{jkl}^{\cdots i}=0$, $B_{jkl}^{\cdots i}=0$.

Hence we choose the above solution $y^a=y^a(x)$ as the coordinate transformation.

Differentiating $P_i^a\,Q_a^j=\delta_i^j$ $\left(Q_a^i=\dfrac{\partial x^j}{\partial y^a}\right)$ by y^a, we have

$$Q_{da}^j=-\,Q_d^i\,Q_c^j\,Q_a^m\,P_{im}^c=-\,Q_d^i\,Q_a^m\,\Gamma^j_{(i)(m)}\,.$$

Differentiating $A_{a(b)}=Q_a^i\,Q_b^j\,A_{i(j)}$ by y^c and using the above relations, we can see

$$\begin{aligned} A_{a(b)c} &=(Q_{ac}^i\,Q_b^j+Q_a^i\,Q_{bc}^j)\,A_{i(j)}+Q_a^i\,Q_b^j\,Q_c^k\,A_{i(j)k} \\ &=Q_a^i\,Q_b^j\,Q_c^k\,\nabla_k\,A_{i(j)} \\ &=0\,, \end{aligned}$$

$$\begin{aligned} B(y,y') &=B(x,x')-A_m\,Q_a^m\,P_{ij}^a\,x'^i\,x'^j \\ &=B(x,x')-A_m\,\Gamma^m_{(i)(j)}\,x'^i\,x'^j \\ &=0\,. \end{aligned}$$

Hence this space be locally flat.

Refferences

1. A. Kawaguchi : Geometry in an n-dimensional space with the arc length $s=\int(A_i\,x''^i+B)^{1\,p}\,dt$, *Trans. of the Amer. Math. Soc.*, 44, (1938).
2. S. Watanabe : On special Kawaguchi spaces, *Tonsor*, 7, (1957).