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A Linear Potential for Quark-Antiquark Pair System

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平野雅宣・池田曜子：クォーク粒子ー反粒子対に対する
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Abstract

We solve the Schroedinger equation of quark-antiquark pair system confined by a linear potential. The exact solution is obtained for L (orbital angular momentum)$=0$. For $L\neq 0$, we use a variational method and obtain the solution approximately. We also present the solution for $L=0$ in the same method to be compared with the exact one.

§ I Introduction

The physics of an elementary particle has changed dramatically during the last four or five years.[1] A new quark carrying a new quantum number called charm has been discovered since the summer of 1974, mostly in $\bar{e}e$ annihilation. There is mounting evidence for the existence of a yet heavier quark.[2]

Therefore all the mesons and baryons could be understood as quark-antiquark and three quark bound states respectively where the quarks come in four varieties at least, or "flavors" called u, d, s and c. It has been established that each quark comes in three so-called "colors", which constructs the quatum chromodynamics (QOD)[3].

On the above conjecture, the spectroscopy of new mesons has been investigated vigorously by assuming the potential between quark and antiquark. For these mesons we deal with a simpler situation because of the non-relativistic treatment as a lowest order approximation. In the static limit, the phenomenological potential between quark and antiquark is suggested to be of the form[4]

$$V(r)=-\frac{4}{3}\alpha_s\frac{1}{r}+\lambda r, \qquad (1)$$

where r is the relative separation of quark and antiquark. The first term is motivated by asymptotic freedom at a short distance and based on QCD[3]. The second term is motivated by quark confinement at a large distance and suggested by the lattice gage theory[5] and the dual string model[6]. The type of the confining potential is also proposed by phenomenological analyses. One is the expected energy ordering[7] of the lowest five states or the Ψ series ;

$$E(1S)<E(1P)<E(2S)<E(1D)<E(2P). \tag{2}$$

Another is the ratio of the $e\bar{e}$ decay width of Ψ and Ψ'[1].

We note that the linear potential strength λ is thought to be independent of the quark mass due to QCD[8] and phenomenologically[9].

In this paper, we investigate the Shroedinger equation considering only the linear confining potential λr of eq. (1). The wave function and energy eigen values are needed in the calculation of the averaged mass level of Ψ series and those mass splitting due to the spin-dependent force shown in the previous papers[10]. Our other purpose is to present the method of solving the problem easily understood by undergraduates.

In § II, we solve the Shroedinger equation for $L=0$ case exactly (L; orbital quantum number). In § III, we present the approximate treatment of the solution for both $L=0$ and $L\neq 0$, which cannot be solved analytically. We also obtain a few expectation values of physical quantities there.

§ II. A linear potential model and $L=0$ spectra

We will solve the Shroedinger equation in the case of a linear potential acting between quark and antiquark. The Hamiltonian of this system is

$$H=\frac{\boldsymbol{p}_1^2}{2m_1}+\frac{\boldsymbol{p}_2^2}{2m_2}+V(|\boldsymbol{r}_1-\boldsymbol{r}_2|)\ , \tag{3}$$

where m_i and $\boldsymbol{r}_i$ are quark or antiquark mass and coordinate respectively. We introduce the relative coordinate ($\boldsymbol{r}$), the center of mass coordinate ($\boldsymbol{R}$) and the momentum $\boldsymbol{p}$, $\boldsymbol{P}$ canonically conjugate to the coordinates respectively as follows,

$$\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2\ ,\qquad \boldsymbol{R}=\frac{m_1\boldsymbol{r}_1+m_2\boldsymbol{r}_2}{m_1+m_2}\ ,$$

$$\boldsymbol{p}=\frac{m_2\boldsymbol{p}_1-m_1\boldsymbol{p}_2}{m_1+m_2},\quad \boldsymbol{P}=\boldsymbol{p}_1+\boldsymbol{p}_2. \tag{4}$$

Then the Hamiltonian is rearranged,

$$H=\frac{\boldsymbol{P}^2}{2(m_1+m_2)}+\frac{\boldsymbol{p}^2}{2\mu}+V(r)\ , \tag{5}$$

where $\mu=m_1m_2/(m_1+m_2)$; reduced mass.
In the center of mass system, we obtain the following Hamiltonian

$$H=\frac{\boldsymbol{p}^2}{2\mu}+V(r)\ . \tag{6}$$

The Shroedinger equation for the Hamiltonian (6) in the spherical polar form is

$$\left\{\frac{-1}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)+V(r)-E\right\}\Psi(\boldsymbol{r})=0\ . \tag{7}$$

Separating angular variables from eq. (7) in the usual way and considering $L=0$ states only, the radial equation

$$\left\{-\frac{1}{2\mu}\frac{d^2}{dr^2}+(\lambda r-E)\right\}R(r)=0\ , \tag{8}$$

where

$$\Psi(\boldsymbol{r})=\frac{R(r)}{r}Y_0^0(\theta,\varphi)\ , \tag{9}$$

and $V(r)=\lambda r$. We note that eq. (8) is the same as the one dimensional Shroedinger equation for a particle in a uniform field (for example, a particle moving in the homogeneous gravitational field or in an accelerating uniform field)[11]. The boundary condition is

$$R(0)=0\ , \tag{10}$$

to be supplemented by

$$R(\infty)\longrightarrow 0\ . \tag{11}$$

Using the abbreviations

$$2\mu\lambda=\frac{1}{l^3}\ ,\qquad 2\mu E=\frac{a}{l^2}\ , \tag{12}$$

and the variable

$$x=\frac{r}{l}-a\ . \tag{13}$$

Equation (8) becomes

$$\frac{d^2u(x)}{dx^2}-xu(x)=0\ , \tag{14}$$

where $R(r)=u(x)$ and the boundary condition

$$u(-a)=0\ ,\qquad u(\infty)\longrightarrow 0\ . \tag{15}$$

If we put

$$u(x)=\sqrt{x}\,f\left(\frac{2}{3}ix^{3/2}\right)\ , \tag{16}$$

by simple calculation, we obtain

$$f''(t)+\frac{1}{t}f'(t)+\left(1-\frac{1}{9t^2}\right)f(t)=0\ , \tag{17}$$

where $t=(2/3)ix^{3/2}$.

The function of $f(t)$ is the Bessel function of 1/3-th order. Therefore the general solution of eq. (14) is expressed as[12]

$$u(x)=\sqrt{x}\left\{c_1J_{1/3}\left(\frac{2}{3}ix^{3/2}\right)+c_2J_{-1/3}\left(\frac{2}{3}ix^{3/2}\right)\right\}\ . \tag{18}$$

In the following, we will try to solve eq. (14) in another way[11]. We assume the following trial solution,

$$u(x)=\int_C e^{xt}f(t)\,dt\ . \tag{19}$$

Using

$$u''(x)=\int_C t^2 e^{xt} f(t)\,dt \ ,$$

$$xu(x)=\int_C \frac{d}{dt}\{e^{xt}f(t)\}\,dt-\int_C e^{xt}f'(t)\,dt \ , \tag{20}$$

and eq. (14), we obtain

$$\int_C e^{xt}\{t^2 f(t)+f'(t)\}\,dt-\int_C \frac{d}{dt}\{e^{xt}f(t)\}\,dt=0 \ . \tag{21}$$

The path of integration c is chosen as

$$\int_C \frac{d}{dt}\{e^{xt}f(t)\}\,dt=0 \ , \tag{22}$$

and from eq. (21) the following is obtained ;

$$t^2 f(t)+f'(t)=0 \ . \tag{23}$$

Then $f(t)$ is

$$f(t)=\text{const.}\times\exp\left(-\frac{t^3}{3}\right). \tag{24}$$

Now the solution of eq. (14) can be written, with the usual normalization as

$$u(x)=-\frac{i}{2\pi}\int_C \exp\left(xt-\frac{t^3}{3}\right)dt \ . \tag{25}$$

The condition, eq. (15) is satisfied and the convergence of the solution is guaranteed if $Re(t^3)>0$ for $|t|\to\infty$ in the complex variable t plane. Then we require

$$\cos 3\varphi>0 \qquad (t=|t|e^{i\varphi}) \ . \tag{26}$$

For example, the good intervals are

$$\left(\frac{\pi}{2},\ \frac{5}{6}\pi\right),\ \left(\frac{7}{6}\pi,\ \frac{3}{2}\pi\right),\ \left(\frac{11}{6}\pi,\ \frac{13}{6}\pi\right) \ . \tag{27}$$

The solution is obtained when we take the path of integration c shown in Fig. 1.

Moving the path c close to the imaginary axes, we obtain the following wave function from eq. (25) substituting t into $i\omega$

$$u(x)=\frac{1}{\pi}\int_0^\infty \cos\left(x\omega+\frac{\omega^3}{3}\right)d\omega \ . \tag{28}$$

This function is defined as the Airy function $A_i(x)$[13, 14];

$$A_i(x)=\frac{1}{\pi}\sqrt{\frac{x}{3}}K_{1/3}(t)$$

$$=\frac{1}{3}\sqrt{x}\{I_{-1/3}(t)-I_{1/3}(t)\} \tag{29}$$

for $x>0$,

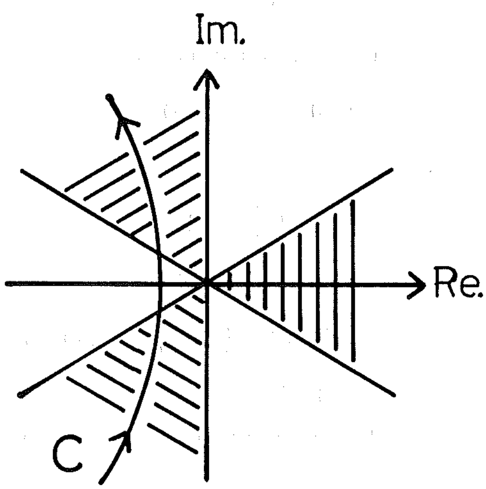

Fig. 1. The path of integration C. The shaded arear is shown in eq.(27).

$$A_i(x)=\frac{1}{3}\sqrt{|x|}\ \{J_{-1/3}(t)+J_{1/3}(t)\} \quad \text{for } x<0\ , \tag{30}$$

where $t=(2/3)|x|^{3/2}$ and $J_m(I_m)$ is the Bessel (modified Bessel) function. The asymptotic behavior follows from the general formula,

$$K_m(x)\longrightarrow\sqrt{\frac{\pi}{2x}}\,e^{-x} \qquad x\longrightarrow\infty\ , \tag{31}$$

so that

$$A_i(x)\longrightarrow\frac{1}{2\sqrt{\pi}}x^{-1/4}\,e^{-2/3x^{3/2}} \qquad x\longrightarrow\infty. \tag{32}$$

According to the boudary condition (15), the Airy function must vanish at $x=-a$,

$$A_i(-a)=0\ . \tag{33}$$

Equations (12) and (33) determine the energy eigen values ;

$$E=\left(\frac{\lambda^2}{2\mu}\right)^{1/3}a_n \tag{34}$$

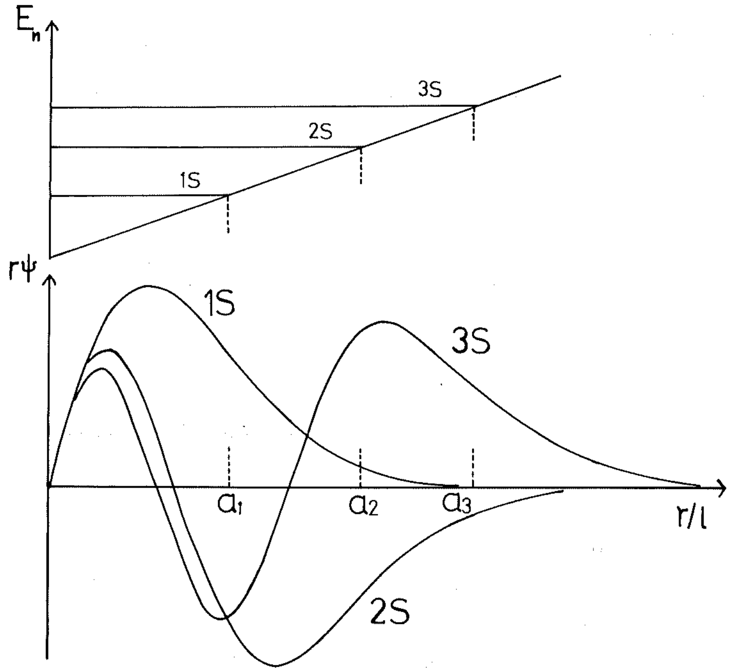

Fig. 2. The energy level and the wave function for $1S$, $2S$ and $3S$ in the linear potential. $r\Psi$ is given in relative units. The dashed line shows the zero of Airy function.

where $a_n(n=1,2\cdots)$ denotes the n-th zeros[13] of the Airy function shown in Table 1.

The total wave function $\Psi(r)$ with the quantum number n is

$$\Psi_n(r)=c\frac{A_i(r/l-a_n)}{r}Y_0^0(\theta,\ \varphi)\ . \tag{35}$$

The constant coefficient c is determined by the normalization as

$$c^{-2}=\int_0^\infty A_i^2(r/l-a_n)\,dr=lA_i'^2(-a_n)\ , \tag{36}$$

where $A_i'(x)=dA_i(x)/dx$ and the last value is obtained by integrating by part two times with $A_i''(x)=xA_i(x)$ and eq. (32). Now we obtain the final expression for $L=0$ wave function,

$$\Psi_n(r)=\frac{1}{A_i'(-a_n)\sqrt{l}}\frac{A_i(r/l-a_n)}{r}Y_0^0(\theta,\ \varphi), \tag{37}$$

and

$$\Psi_n(\vec{0})=\left(\frac{\mu\lambda}{2\pi}\right)^{1/2}, \tag{38}$$

where $l=(2\mu\lambda)^{-1/3}$. We plot, for example, 1S, 2S and 3S wave function with these energy levels in Fig.2.

Table 1. The n-th zero (a_n) of the Airy function.[13]

n	a_n
1	2.338
2	4.088
3	5.521
4	6.787
5	7.944
6	9.023
7	10.040
8	11.009
9	11.936
10	12.829

§ III. The variational method and approximating harmonic oscillator potential

We have found the exact solution of eq.(7) for $L=0$ case in the previous section. For $L\neq 0$ case, it is hard to solve the Shroedinger equation analytically. Therefore we will express the $L\neq 0$ wave function in terms of the simple analytic wave function approximately and obtain these energy expectation values. We also need the wave function to calculate the following exepectation value i.e. ⟨r⟩ , ⟨1/r⟩ ⋯.

For the above purpose, we introduce the harmonic oscillator potential[15] into the Hamiltonian as

$$H=\left(\frac{\boldsymbol{p}^2}{2\mu}+\frac{k}{2}\boldsymbol{r}^2\right)+\left(\lambda r-\frac{k}{2}\boldsymbol{r}^2\right)\equiv H_0+H_1. \tag{39}$$

We treat H_1 in the first order perturbation theory and obtain the approximated energy levels,

$$E_n^{approx}=\langle\Psi_n|H|\Psi_n\rangle, \tag{40}$$

where Ψ_n is the n-th harmonic oscillator eigen function belonging to H_0. The harmonic oscillator coupling constant k is chosen so that the energy E_n^{approx} is minimized. In other words, we will express the wave function for any L case in terms of a single harmonic oscillator wave function approximately, using a variational method to select the optimum coupling constant k.

The energy levels for the Hamiltonian H_0 occur for $E=(L+2p+3/2)\omega$, $(\omega=\sqrt{k/\mu})$, where L is the orbital angular momentun quantum number and $p=0,1,2\cdots$ is associated with the number of nodes in th radial wave function. The corresponding wave function can be written as

$$\Psi_{nLm}=N(\alpha r)^L \mathcal{L}_p^{L+1/2}(\alpha^2 r^2) exp(-\frac{1}{2}\alpha^2 r^2)\, Y_L^m(\theta. \varphi), \tag{41}$$

where $n=L+p+1$(the principle quantum number), $\alpha^2=\mu\omega=\sqrt{\mu k}$ and $\mathcal{L}$ is a Laguerre polynomial. The normralization constant N is given by

$$N^2=\frac{2\,\alpha^3}{\sqrt{\pi}(p+L+\frac{1}{2})(p+L-\frac{1}{2})\cdots\frac{3}{2}\times\frac{1}{2}\times p!}. \tag{42}$$

A few of these functions which are denoted by the combination (n,L) are written down

$$\Psi_{1S}=\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{1/2} exp(-\frac{1}{2}\alpha^2 r^2)\, Y_0(\theta,\varphi),$$
$$\Psi_{1P}=\sqrt{\frac{2}{3}}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{1/2} \alpha r\, exp(-\frac{1}{2}\alpha^2 r^2)\, Y_1(\theta. \varphi),$$
$$\Psi_{1D}=\sqrt{\frac{4}{15}}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{1/2} \alpha^2 r^2\, exp(-\frac{1}{2}\alpha^2 r^2)\, Y_2(\theta,\varphi),$$
$$\Psi_{2S}=\sqrt{\frac{2}{3}}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{1/2}(\frac{3}{2}-\alpha^2 r^2)\, exp(-\frac{1}{2}\alpha^2 r^2)\, Y_0(\theta,\varphi),$$
$$\Psi_{3S}=\sqrt{\frac{2}{15}}\left(\frac{4\alpha^3}{\sqrt{\pi}}\right)^{1/2}(\frac{15}{4}-5\alpha^2 r^2+\alpha^4 r^4)\, exp(-\frac{1}{2}\alpha^2 r^2)\, Y_0(\theta,\varphi). \tag{43}$$

We treat, for example, 1S state (the ground state) accoding to the above procedure. A straightforward calculation gives the approximated 1S state energy

$$E_{1S}^{approx}=\frac{3}{4}\omega+\langle\Psi_{1S}|\lambda r|\Psi_{1S}\rangle$$
$$=\frac{3}{4}\omega+\frac{2}{\sqrt{\pi}\,\alpha}\lambda. \tag{44}$$

Minimizing this with respect to k yields

$$\omega=\sqrt{\frac{k}{\mu}}=(\frac{16}{9\pi}\frac{\lambda^2}{\mu})^{1/3}, \tag{45}$$
$$\alpha=(\mu k)^{1/4}=\left(\frac{4}{3\sqrt{\pi}}\lambda\mu\right)^{1/3}. \tag{46}$$

Substituting eqs. (45) and (46) into eq. (44), we obtain

$$E_{1S}==\left\{\frac{3}{4}\left(\frac{32}{9\pi}\right)^{1/3}+2\left(\frac{3}{2\pi}\right)^{1/3}\right\}\left(\frac{\lambda^2}{2\mu}\right)^{1/3},$$
$$\Psi_{1S}==\left(\frac{4}{\sqrt{\pi}}\right)^{1/2}\left(\frac{4}{3\sqrt{\pi}}\lambda\mu\right)^{1/2} exp\left\{-\frac{1}{2}\left(\frac{4}{3\sqrt{\pi}}\lambda\mu\right)^{2/3} r^2\right\} Y_0^0(\theta,\varphi),$$
$$\Psi_{1S}(0)=\left(\frac{4}{3\pi^2}\lambda\mu\right)^{1/2}. \tag{47}$$

When we use the variational method to the other states in the same way, we can fix the corresponding optimum coupling constant k (α,ω) for each state. We summarize the minimized energies and α values as follows,

$$E_{1S}=\left\{\frac{3}{4}\left(\frac{32}{9\pi}\right)^{1/3}+2\left(\frac{3}{2\pi}\right)^{1/3}\right\}\left(\frac{\lambda^2}{2\mu}\right)^{1/3},$$

$$E_{2S}=\left\{\frac{7}{4}\left(\frac{72}{49\pi}\right)^{1/3}+2\left(\frac{63}{8\pi}\right)^{1/3}\right\}\left(\frac{\lambda^2}{2\mu}\right)^{1/3},$$

$$E_{3S}=\left\{\frac{11}{4}\left(\frac{225}{242\pi}\right)^{1/3}+2\left(\frac{2475}{128\pi}\right)^{1/3}\right\}\left(\frac{\lambda^2}{2\mu}\right)^{1/3},$$

$$E_{1P}=\left\{\frac{5}{4}\left(\frac{512}{225\pi}\right)^{1/3}+2\left(\frac{40}{9\pi}\right)^{1/3}\right\}\left(\frac{\lambda^2}{2\mu}\right)^{1/3},$$

$$E_{1D}=\left\{\frac{7}{4}\left(\frac{2048}{1225\pi}\right)^{1/3}+2\left(\frac{224}{25\pi}\right)^{1/3}\right\}\left(\frac{\lambda^2}{2\mu}\right)^{1/3}, \qquad (48)$$

and

$$\alpha(1S)=\left(\frac{4}{3\sqrt{\pi}}\lambda\mu\right)^{1/3},$$

$$\alpha(2S)=\left(\frac{6}{7\sqrt{\pi}}\lambda\mu\right)^{1/3},$$

$$\alpha(3S)=\left(\frac{15}{22\sqrt{\pi}}\lambda\mu\right)^{1/3},$$

$$\alpha(1P)=\left(\frac{16}{15\sqrt{\pi}}\lambda\mu\right)^{1/3},$$

$$\alpha(1D)=\left(\frac{32}{35\sqrt{\pi}}\lambda\mu\right)^{1/3}, \qquad (49)$$

These energies are expressed formally

$$E_{nL}=\left(\frac{\lambda^2}{2\mu}\right)^{1/3}\varepsilon_{nL} \qquad (50)$$

We compare this with the exact value of eq.(34) for $L=0$ state in Table 2, where 1P and 1D states are also listed. The approximation for the level energy is very good.

Table 2. The comparison of a_n with obtained ε_{nL} for the lowest levels.

$Level$	a_n	ε_{nL}	$Error\%$
$1S$	2.338	2.345	0.3
$2S$	4.088	4.075	− 0.3
$3S$	5.521	5.498	− 0.4
$1P$		3.368	
$1D$		4.254	

For many purposes, the wave function at the origin is used. So its replacement by the harmonic oscillator wave function is important. From eqs.(38) and (47) we find

$$\Psi_{1S}^{approx}(0)/\Psi_{1S}(0)=\left(\frac{8}{3\pi}\right)^{1/2}\fallingdotseq 0.92, \qquad (51)$$

showing that the approximation is good to 8% even in zeroth order.

We next present the expectation value $\langle 1/r\rangle$, $\langle 1/r^3\rangle$ and $\sqrt{\langle r^2\rangle}$, calculated by the wave function of eq. (43) and eq.(49). These were used in the previous papers[10] or will be used in the forthcoming paper. They are

$$\langle 1S|\frac{1}{r}|1S\rangle=\frac{2}{\sqrt{\pi}}\alpha(1S)=\left(\frac{32}{3\pi^2}\lambda\mu\right)^{1/3},$$

$$\langle 2S|\frac{1}{r}|2S\rangle=\frac{5}{3\sqrt{\pi}}\alpha(2S)=\frac{5}{3}\left(\frac{6}{7\pi^2}\lambda\mu\right)^{1/3},$$

$$\langle 3S|\frac{1}{r}|3S\rangle=\frac{89}{60\sqrt{\pi}}\alpha(3S)=\frac{89}{60}\left(\frac{15}{22\pi^2}\lambda\mu\right)^{1/3},$$

$$\langle 1P|\frac{1}{r}|1P\rangle=\frac{4}{3\sqrt{\pi}}\alpha(1P)=\frac{4}{3}\left(\frac{16}{15\pi^2}\lambda\mu\right)^{1/3},$$

$$\langle 1D|\frac{1}{r}|1D\rangle=\frac{16}{15\sqrt{\pi}}\alpha(1D)=\frac{16}{15}\left(\frac{32}{35\pi^2}\lambda\mu\right)^{1/3},$$

$$\langle 1P|\frac{1}{r^3}|1P\rangle=\frac{4}{3\sqrt{\pi}}\alpha(1P)^3=\frac{64}{45\pi}\lambda\mu,$$

$$\langle 1D|\frac{1}{r^3}|1D\rangle=\frac{16}{15\sqrt{\pi}}\alpha(1D)^3=\frac{512}{525\pi}\lambda\mu,$$

$$\langle 1S|r^2|1S\rangle^{1/2}=\sqrt{\frac{3}{2}}\,\frac{1}{\alpha(1S)}=\sqrt{\frac{3}{2}}\left(\frac{3\sqrt{\pi}}{4}\,\frac{1}{\lambda\mu}\right)^{1/3}. \qquad (52)$$

In the last, we mention that the reason why this simple replacement of the linear potential by a harmonic oscillator potential works so well is easily understood. For large r, they behave in the very different way, but the wave function falls off exponentially so that this region is of no importance. For small r, the boundary condition $R(0)=0$ in eq.(10) suppresses this region though the potentials are also different. Therefore, the region where the wave functions for both potentials behave rather similarly, contributes to the calculation of the expectation value.

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