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On Some Open Problems in the Theory of Quasiconformal Mappings II (Area Distortion under Quasiconformal Mappings)

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岡部　勝幸：疑等角写像の理論におけるいくらかの未解決の問題についてII
（疑等角写像による測度の歪みについて）

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Abstract

Let f be a plane K-quasiconformal mapping of the unit disc U onto itself, normalized by $f(0)=0$, and m be the 2-dimensional Lebesque measure. Gehring and Reich [12] showed that there exist a constant a and a function $b(K)$, where $1 \leq a \leq 40$, $b(K)>0$ and $b(K)=1+0(K-1)$ as $K \longrightarrow 1$, such that

$$\frac{m(f(E))}{\pi} \leq b(K)\left(\frac{m(E)}{\pi}\right)^{K-a}$$

for each measurabl set $E \subseteqq U$. They also conjectured that the above theorem holds with $a=1$. However, unfortunately they could not prove it in [12]. After that, this problem was posed by Gehring at the Romanian-Finnish Seminar on Teichmüller spaces and quasiconformal mappings. (See [13].) As far as the author knows, this problem is still open. In this paper, we shall give an answer to this problem.

1. Introduction

Throughout this paper, by K-quasiconformality we mean K-quasiconformality according to Gehring's analtytic definition.

The relations among the definitions of K-quasiconformality are given in Caraman's paper [5] and book [6], and the notations and terminology which are used in this paper are given in [6], [18], [20] and [27]. It is well known that q. c. mappings preserve sets of measure zero. "In connection with deeper studies of the degree of regularity of q. c. mappings, it is of interest to investigate to what extent one can say that $m(f(E))$ is small whenever $m(E)$ is small".

—— Gehring and Reich [12]. We shall enumerate some previous works on this problen. Using a theorem on L^p-integrability of the derivatives of q. c. mappings, Bojarski established the following theorem on absolute continuity with respect to the area measure. There exist a pair of functions of K, $a(K)>0$ and $b(K)$, such that if f is a K-q. c. mapping of the unit disc U onto itself with $f(0)=0$, then

$$\frac{m(f(E))}{\pi} \leq b(K)\left(\frac{m(E)}{\pi}\right)^{a(K)}$$

for each measurable set $E \subseteqq U$. (See Ahlfors [2] and Lehto-Virtanen [20].) Successively Lehto [19] studied the integrability question for q. c. mappings. His results imply that $a(K)=K^{-a}$ and a is a constant, $a>1$. For general n, a similar result has been given by Gehring [11]. Furthermore, Gehring and Reich [12] showed that $1 \leq a \leq 40$, $b(K)>0$ and $b(K)=1+0(K-1)$ as $K \longrightarrow 1$. Their proof consists of using the parametric representation for a q. c. mapping and the two dimensional Hilbert transformations. Using a similar method as in [12], Reich [24] proved that $1 \leq a \leq 20$. (It was reporeted that $1 \leq a \leq 17$, because of a computational error. See Introduction of [16].) On the other hand, similar results for a hyperbolic area have been given by Gehring [10] and Kelingos [16]. If we restrict ourselves to the case when $n=2$, then $a(K)=1/K$ is an immediate consequence of Theorem 2 [22, p. 68] and the results due to Lohto [19]. However, for higher dimensions the corresponding results to Lehto's have been unknown and moreover, this method does not give much information about the function $b(K)$. Thus the above method can not be used. In the present paper we shall give an answer to this problem, and its proof is based mainly on the Hölder continuity of q. c. mappings.

2. The Grötzsh and the Teichmüller rings

2. 1 By a ring A is meant a domain in euclidean n-space R^n whose complement consists of two components C_0 and C_1, where C_0 is bounded. For each $0<r<1$, the Grötzsh ring $A_G(r)$ is the ring whose complemementary components are $C_0=\{(x_1, \cdots\cdots\cdots x_n): 0 \leq x_1 \leq r, x_j=0, 2 \leq j \leq n\}$ and $C_1=\{(x_1, \cdots\cdots x_n): \sum_{j=1}^{n} x_j^2 \leq 1\}$. Simillarly for each $r>0$, the Teichmüller ring $A_T(r)$ is the ring whose complementary components are $C_0=\{(x_1, \cdots\cdots, x_n): -1 \leq x_1 \leq 0, x_j=0, 2 \leq j \leq n\}$ and $C_1=\{(x_1, \cdots\cdots, x_n): x_1 \geq r, x_j=0, 2 \leq j \leq n\}$. It is known that the modules of these rings have the following properties. If, following Teichmüller, we set mod $A_G(r)=\log \Phi(r)$ and mod $A_T(r)=\log \Psi(r)$, then

(2. 1) $$\Psi(r)=\Phi[1/(1+r)^{1/2}]^2$$

Moreover $r\Phi(r)$ is non-increasing in (0, 1) and there exist constants λ_n such that

(2. 2) $$1<r\Phi(r)<\lambda_n \text{ for } 0<r<1$$

and

(2. 3) $$\lim_{r \to 0} r\Phi(r)=\lambda_n.$$

Unfortunately the values of the constants λ_n are unknown except when $n=2$, in which case $\lambda_2=4$. In a recent paper [3], Anderson showed that $4 \leq \lambda_n \leq e^n$ (cf. [5], [8] and [10]) and $\lim_{n \to \infty} \lambda_n^{1/n}=e$.

2. 2 Following Hersh and Pfluger, we set $\mu(r)=\log\Phi(r)$ and $\varphi_K(r)=\mu^{-1}(\mu(r)/K)$ for $K>1$. It is then easy to see that the functions $\mu(r)$ and $\varphi_K(r)$ have the same properties as in the case when $n=2$. Here we obtain, by (2. 1), (2. 2) and (2. 3),

$$\varphi_K(r)<\lambda_n^{1-1/K}\,r^{1/K} \tag{2. 4}$$

and

$$\lim_{r\to 0}\varphi_K(r)/r^{1/K}=\lambda_n^{1-1/K}. \tag{2. 5}$$

Now suppose that f is a K-q. c. mapping of the unit ball $B(0,1)$ into itself, normalized by $f(0)=0$. Then (2. 4) and (2. 5) imply that

$$|f(x)|\leq\varphi_K(|x|) \tag{2. 6}$$

and

$$\limsup_{|X|\to 0}\frac{|f(x)|}{|x|^{1/K}}\leqq\lambda_n^{1-1/K}. \tag{2. 7}$$

2. 3 Next suppose that A is a ring whose complementary components $C_0=\{(x_1,\cdots\cdots,x_n):|x|\leq x_1\leq 1/|x|,\ x_j=0,\ 2\leq j\leq n,\ 0<|x|<1\}$ and $C_1=\{(x_1,\cdots\cdots,x_n):x_1\leq 0,\ x_j=0,\ 2\leq j\leq n\}$, and that f is a K-q. c. mapping of the unit ball onto itself, nomalized by $f(0)=0$. Since f can be extended to a K-q. c. mapping of the whole space onto itself by means of the reflection principle (Theorem 2 [6, p. 381]), we have (mod $A)/K\leq$mod $f(A)$. By using a suitable Möbius transformation, we obtain mod $A=$mod $A_T(|x|^2/(1-|x|^2))=\mu((1-|x|^2)^{1/2})$. Furthermore, by means of the spherical symmetrization of $f(A)$ (cf. [7]), we have mod $f(A)\leq\mu((1-|f(x)|^2)^{1/2})$. we thus conclude that

$$\frac{1-|f(x)|^2}{(1-|x|^2)^{1/K}}\leq\lambda_n^{2(1-1/K)}. \tag{2. 8}$$

Remarks. (1) A similar result to (2. 6) has been given by Väisälä [26, Theorem 8. 3].
(2) If $K=1$ and $n=2$, then the formula (2. 7) reduces to Schwarz's lemma. Other formulas of the Schwarz type have been given by Ikoma [14] and Kiernan [17].
(3) (2. 8) is an n-dimensional version of a result in [18, p. 116].

3. Distortion of euclidean distance under quasiconformal mappings of a ball

We consider n-dimensional K-q. c. mappings of the unit ball B onto itself, normalized by $f(0)=0$. If we set

$$C(K)=\sup\frac{|f(x)-f(y)|}{|x-y|^{1/K}}, \tag{3. 1}$$

where the supremum is taken over all pairs of points x, y, $x\neq y$ with $|x|\leq 1$, $|y|\leq 1$ and all K-q. c. mappings, then it is well known that there exists a constant C such that $1\leq C(K)\leq C$. We can take $C=16$ for $n=2$ and $C=4\lambda_n^2$ for general n. These facts can be found, for instance, in the papers of Ahlfors [1], Callender [4], Gehring [8], Mori [21] and reŝentyark [25]. Now we wish to show that $C(K)=1+0(K-1)$ *as* $K\longrightarrow 1$. Later we will use it to make an estimation of $b(K)$. For this purpose we need the following theorem.

Theorem 3. 1. Let f be a K-quasiconformal mapping of the unit ball onto itself, normalized by $f(0)=0$. Then

$$\lim_{r\to 0}\frac{L(x,f,r)}{r^{1/K}}\leq\lambda_n^{1-1/K}\frac{(1-|f(x)|^2)}{(1-|x|^2)^{1/K}} \tag{3. 2}$$

where $L(x,f,r)=\max_{|x-y|=r}|f(x)-f(y)|$.

Proof. In order to estimate the distance between the images of two given points x_1, x_2, we firs carry x_1 and $f(x_1)$ onto the origin by the Möbius transformations g_1 and g_2,

$$g_1(x)=(1-\frac{1}{|x_1|})\frac{x-\frac{x_1}{|x_1|^2}}{\left|x-\frac{x_1}{|x_1|^2}\right|^2}-\frac{x_1}{|x_1|^2}$$

and

$$g_2(x)=(1-\frac{1}{|f(x_1)|^2})\frac{x-\frac{f(x_1)}{|f(x_1)|^2}}{\left|x-\frac{f(x_1)}{|f(x_1)|^2}\right|^2}-\frac{f(x_1)}{|f(x_1)|^2}$$

The conposition $h=g_2\,f\,g_1^{-1}$ then maps the unit ball B onto with the origin left in variant (2. 6) and (2. 7) this yield the tollowing result. Fovevery paiv of point x_1, x_2, we have

$$\frac{|f(x_1)-f(x_2)|}{|x_1-x_2|^{1/K}}\leq\lambda_n^{1-1/K}\frac{|f(x_1)|\left|f(x_2)-\frac{f(x_1)}{|f(x_1)|^2}\right|}{\left\{|x_1|\left|x_2-\frac{x_1}{|x_1|^2}\right|\right\}^{1/K}}.$$

Next we set $x_1=x$, $x_2=y$, and we obtain (3. 2) by letting $y\to x$. Thus the proof is complete.

Now we shall prove that $C(K)=1+0(K-1)$ as $K\to 1$. Though this seems clear at a glance, it is not very evident to the author. As far as the author knows, this has not been stated certainly.

Lemma 3. 2. $C(K)=1+0(K-1)$ as $K\to 1$.

Proof. For every pair of points x, y, $x\neq y$ with $|x|\leq 1$, $|y|\leq 1$, we set

$$C(K,\ x,\ y)=\sup\frac{|f(x)-f(y)|}{|x-y|^{1/K}}$$

where the supremun is taken over all K-q. c. mappings. It is easy to see that $C(K,\ x,\ y)>1$. By the theory of the normal family, there exists a K-q. c. mapping f_0 such that

$C(K,\ x,\ y)=\frac{|f_0(x)-f_0(y)|}{|x-y|^{1/K}}$ for each pair of given points x, y, $x\neq y$. On the other hand, Theorem 1 [6, p. 358] implies that there a function $\sigma(K)$, $\sigma(K)\to 0$ as $K\to 1$ so that for each

f_0, we may find a conformal mapping (a rotation by Reade's theorem [6, p. 323]) $M(x)$ so that $|f_0(x)-M(x)|<\sigma(K)$ for all x with $|x|\leq 1$. Here $C(K, x, y)\to 1$ as $K\to 1$ for each given point $x, y, x\neq y$. However this depends on x and y. Theorem 3. 1 and (2. 8) imply that

$$\overline{\lim_{r\to 0}}\frac{L(x, f_0, r)}{r^{1/K}}\leq \lambda_n^{3(1-1/K)} \text{ for each } f_0.$$

From this inequality, it follows that $\sup_{x\neq y} C(K, x, y)=1+0(K-1)$ as $K\to 1$. Since $C(K)=\sup_{x\neq y} C(K, x, y)$, we thus conclude that $C(K)=1+0(K-1)$ as $K\to 1$.

Remarks. (1) It is conjectured that $C(K)=16^{1-1/K}$ for $n=2$ by Lehto and Virtanen [20, p. 68].
(2) If $K=1$ and $n=2$, then Theorem 3. 1 reduces to the classical Schwarz- Pick lemma.

On the other hand, some generalizations for K-q. c. mappings of the Koebe distortion formula and Koebe's onequarter theorem have been given by Gehring [8], Juve [15] and Pfluger [23].

(3) For the details of the normal family, we refer to Gehring [9], Lehto-Virtanen [20] and Väisälä [26], [27].

4. Area distortion under quasiconformal mappings

We shall show a slightly more precise form of the result due to Gehring-Reich [12, Theorem 1]. We denote by $B=B(x, r)$ and by $m=m_x$ the open n-ball of radius r about x and the n-dimensional Lebesque measure in R^n.

4. 1 First we begin by proving the following almost clear lemma on the property of α-Riesz potential.

Lemma 4. 1. Let E be an arbitary measurable set contained in $B(0, 1)$. Then, for each $0<\alpha<n$,

$$\int_E \frac{dm_y}{|x-y|^{n-\alpha}}\leq \frac{n}{\alpha}\Omega_n^{1-\alpha/n}\ m(E)^{\alpha/n} \text{ for all } x\in R^n,$$

where $\Omega_n=m(B(0, 1))$.

Proof. We may choose a positive number r so that $m(E)=m(B(x, r))$ for every $x\in R^n$. Since

$$\int_{B(x, r)-E}\frac{dm_y}{|x-y|^{n-\alpha}}\geq \int_{E-B(x, r)}\frac{dm_y}{|x-y|^{n-\alpha}},$$

we obtain easily that

$$\int_E\frac{dm_y}{|x-y|^{n-\alpha}}\leq \int_{B(x, r)}\frac{dm_y}{|x-y|^{n-\alpha}}\leq \frac{n}{\alpha}\Omega_n^{1-\alpha/n}\, m(E)^{\alpha/n},$$

which completes the proof.

Theorem 4. 2. There exists a function $b(K)$ where $b(K)>0$, and $b(K)=1+0(K-1)$ as $K\to 1$, such that if f is an n-dimensional K-quasiconformal mapping, $n\geq 2$, of the unit ball onto itself with $f(0)=0$, then

(4. 1) $$\frac{m(f(E))}{\Omega_n} \leq b(K)\left(\frac{m(E)}{\Omega_n}\right)^{1/K}$$

for each measurbale set $E \subseteq B(0, 1)$. The exponent $1/K$ is sharp.

Proof. Let E be an arbitary measurable subset of $B(0, 1)$. By Hölder-continuity of f, we can then define $G(y)$ by

(4. 2) $$G(y)=\left[\int_E\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^n dm_x\right]^{1/n}$$

for each $y \in E$. Lemma 4. 1 and (3. 1) imply that

(4. 3) $$G(y)^n \leq KC(K)^n \Omega^{1-1/K} m(E)^{1/K} \text{ for all } y \in E.$$

Next we consider the function F_h.

$$F_h(x)=\frac{1}{m(B(x, h))}\int_{B(x,h)}\frac{|f(x)-f(y)|}{|x-y|}dm_y$$

for each point $x \in E$ and for each positive number h. The function F_h is meaningful, because f can be extended to a K-q. c. mapping of the whole space R^n onto itself by means of the reflection principle for q. c. mappings. For convenience we write $B=B(x, h)$. From Minkowski's inequality (for instance, [6, p. 109]), it follows that

(4. 4) $$\begin{aligned}\int_E F_h(x)^n dm_x &= \int_E\left[\frac{1}{m(B)}\int_B\frac{|f(x)-f(y)|}{|x-y|}dm_y\right]^n dm_x\\ &\leq\left\{\frac{1}{m(B)}\int_B\left[\int_E\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^n dm_x\right]^{1/n}dm_y\right\}^n\\ &\leq\left[\frac{1}{m(B)}\int_B G(y)\,dm_y\right]^n.\end{aligned}$$

In conjection with (4. 3), (4. 4) yields

(4. 5) $$\int_E F_h(x)^n dm_x \leq KC(K)^n \Omega_n^{1-1/K} m(E)^{1/K}$$

for each $h>0$. On the other hand, since $J_f(x) \leq K^{n-1} l_f^n(x)$ a. e., where J_f, l_f denotes the jacobian of f, the minimal stretching of f respectively, by the same argument as in [22, p. 67], we have

(4. 6) $$\begin{aligned}m(f(E)) &= \int_E J_f(x)\,dm_x\\ &\leq K^{n-1}\int_E l_f(x)^n dm_x \leq K^{n-1}\varliminf_{h\to 0}\int_E F_h(x)^n dm_x.\end{aligned}$$

By (4. 5) and (4. 6), we conclude that

$$\frac{m(f(E))}{\Omega_n} \leq K^n C(K)^n \left(\frac{m(E)}{\Omega^n}\right)^{1/K}.$$

(4. 1) thus holds with $b(K)=K^nC(K)^n$. Moreover, Lemma 3. 2 implies that $b(K)$ is of the from $1+0(K-1)$ as $K\to1$. Finally the mapping $g(x)=|x|^{a-1}x$, $a=1/K$ shows that the exponent $1/K$ is sharp. This completes the proof.

4. 2 We conclude this paper with three easy applications of Theorem 4. 2. We omit their proofs and details, because these results are immediately followed from combining Theorem 4. 2 and the results established by Gehring-Reich [12, Theorem 3] and Kelingos [16, Section 8]. The first of these sharpens Theorem 2 [12] and Theorem 3 [24].

Theorem 4. 3. (Gehring-Reich [12], Reich [24]). Let $\tilde{\chi}_E$ denote the Hilbert transform of χ_E which is the characteristic function of E and let U denote the unit disc. Then there exists a universal constant b such that

$$\int_U |\tilde{\chi}_E|\,dm \leq m(E)\log\frac{1}{m(E)} + bm(E)$$

for the mesurable set $E \supseteq U$.

The second and third applications are concerned with a hyperbolic area under q. c. mappings. We denote by $R_h(E)$ the hyperbolic radius of E defined to be the hyperbolic radius of the smallest hyperbolic disc containing E. Then we can state slightly more precise forms of Theorem 1 and Theorem 2 [16].

Theorem 4. 4. (Kelingos [16]). For each R, $0<R<\infty$, there exists an increasing continuous function $\psi_{K,R}(t)$, $0<t<\infty$, such that if f is a plane K-quasiconformal mapping of a domain D, then

$$\frac{m_h(f(E))}{\pi} \leq \psi_{K,R}\left(\frac{m_h(E)}{\pi}\right)$$

for each measurable set $E\subset D$ with $R_h(E)\leq R$. Furthermore $\psi_{K,R}(t)=b_1(K,\ R)t^{1/K}$ for $0<t<t_0<1$ and $\psi_{K,R}(t)=b_2(K,\ R)Kt$ for $t>1$, where each $b_i(K,\ R)$ is strictly greater than 0 and is equal to $1+0(K-1)$ as $K\to1$.

Theorem 5. 5 (Kelingos [16]). For each r, $0<r<1$, there exists a constant $c(r)>1$ such that for each measurable set E contained in the disc $|z|\leq r$,

$$\int_U |h_E(z)|\,d\sigma \leq \begin{cases} m_h(E)\log\dfrac{\pi}{m_h(E)} + cm_h(E) & \text{if } m_h(E)\leq\pi \\ cm_h(E) & \text{if } m_h(E)\leq\pi \end{cases}$$

where $h_E(z)$ is the Hilbert transform of the hyperbolic characteristic function of a subset E of the unit disc U defind by $h_E(z)=\chi_E(z)/(1-z\bar{z})^2$.

Remarks. (1) In [16, p. 134]), the functions $\psi_{K,R}(t)$, $b_i(K,\ R)$ are materialized. Theorem 4. 4 holds the functions given by replacing $a=20$ by $a=1$.
(2) A distortion theorem concerning the hyperbolic area of discs under q. c. mappings has been given by Gehring [10, Theorem 3].

References

[1] Ahlfors, L., (1954), On quasiconformal mappings. J. Analyse Math., 3. p. 1-58.

[2] ———, (1966), Lectures on quasiconformal mappings. Van Nostrand, Princeton, New Jersey. p. 146.

[3] Anderson, G. D., (1976), Dependence on dimension of a constant related to the Grötsch ring. Proc. Amer. Math. Soc. P. 77-80.

[4] Callender, E. D., (1960), Hölder continuity of N-dimensional quasiconformal mappings. Pacific. J. Math. 10. p. 499-516.

[5] Caraman, P., (1967), On the equivalence of the definitions of the n-dimensional quasiconformal homeomorphisms (QCfH). Rev. Roumaine Math. Pures Appl. 12. p. 887-941.

[6] ———, (1974), n-Dimensional quasiconformal mappings. Abacus Press, Tunbridge Wells, Kent. p. 551.

[7] Gehring, F. W., (1961), Symmetrization of rings in space. Trans. Amer. Math. Soc. 101. P. 499-519

[8] ———, (1962, Rings and quasiconformal mappings in space. Trans. Amer. Math. Soc. 103. P. 353-393.

[9] ———, (1963), The Caratheodory convergence theorem for quasiconformal mappings in space. Ann. Acad. Sci. Fenn. Ser. A. I. 336/11, p. 1-21.

[10] ———, (1971), Inequalities for condensers, hyperbolic capacity, and extremal lengths. Michigan. Math. J., 18. p. 1-20.

[11] ———, (1973), The L^p-integrability of the partial derivatives of a quasiconfomal mapping. Acta. Math. 130. p. 265-277.

[12] Gehring, W. F. and Reich, E., (1966), Area distortion under quasiconformal mappings. Ann. Acad. Sci. Fenn. AI. 388. p. 1-15.

[13] Hitotumatu, S., (1970), The seminar on quasiconformal mappings in Rumania. Sugaku. 22-2. p. 150-151.

[14] Ikoma, K., (1964), On schwartz lemma for K-quasiconformal mappings in space. Sugaku. 16-2. p. 104-106.

[15] Juve, Y., (1965), Über gewisse Verzerrungseigenschften Konformer und quasikonformer Abbildungen. Ann. Acad. Sci. Fenn. AI. 174. p. 1-40.

[16] Kelingos, J. A., (1972), Distortion of hyperbolic area under quasiconformal mappings. Notices Amer. Math. Soc. 19. A-396. p. 127-139.

[17] Kiernan, J. P., (1979), Quasiconformal mappings and Schwarz's lemma. Trans. Amer. Math. Soc. 148. p. 185-197.

[18] Künzi, H. P., (1960), Quasikonforme Abbildungen. Berlin, Göttingen, Heidelberg. Springer-Verlag. p. 182.

[19] Lehto, O., (1965), Remarks on the integrability of the deivatives of quasiconformal mappings. Ann. Acad. Sci. Fenn. Ser. A. 371. p. 1-8.

[20] Lehto, O. and Virtanen, K. I., (1973), Quasiconformal mappings in the plane. Springer-Verlag. Berlin. Heidelberg, New York. p. 258.

[21] Mori, A., (1956), On an absolute constant in the theory of quasiconformal mappings. J. Math. Soc. Japan. 8. p. 156-166.

[22] Okabe, K., (1977), On some open problems in the theory of quasiconformal mappings. I. (The L^p-integrability of the partial derivatives of a quasiconformal mapping). J, Hokkaido Univ. of Education. Sec-IIA. 28-2. p. 65-69.

[23] Pfluger, A., (1951), Quasikonforme Abbildungen und logarithmische Kapazität. Ann. Inst. Fourier. Grenoble 2, p. 69-80.

[24] Reich, E., (1967), Some estimates for the two-dimensional Hilbert transform. J. Analyse Math., Vol. 18. p. 279-293.

[25] Resetnyak, Yu. G., (1960), A sufficient condition for Hölder continuity of a mapping. Soviet Math. 1. p. 76-78.

[26] Väisälä, J., (1961), On quasiconformal mappings in space. Ann. Acad. Sci. Fenn. AI. 298. p. 1-36.

[27] ———, (1971), Lectures on n-dimensional quasiconformal mappings. Springer-Verlag. Berlin, Heidelberg. New York, p. 144.