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Remarks on Kaehlerian Manifolds with Vanishing Bochner Curvature Tensor

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長谷川和泉・中根　敏幸：Bochner曲率テンソルが消えるケーラー多様体について

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Abstract

Let M be a Kaehlerian manifold and let $B=(B_{hijk})$ be its Bochner curvature tensor. If M is of constant holomorphic sectional curvature, we have

(＊)　　　$B=0.$

There is a question: Under what additional condition does this condition (＊) imply that M is of constant holomorphic sectional curvature ?

In this paper, we offer to prove the best possible pinching theorem on the length of the Ricci tensor for Bochner-Kaehlerian manifold (i.e., Kaehlerian manifold satisfying the condition (＊)) with constant scalar curvature to be of constant holomorphic sectional curvature.

§ 1. Statement of results

Recently, S. I. Goldberg and M. Okumura [3] proved

THEOREM A. *Let M be an n-dimensional compact conformally flat Riemannian manifold with constant scalar curvature R. If the length of the Ricci tensor is less than $R/\sqrt{n-1}$, $n \geqq 3$, then M is a space of constant curvature.*

For a Kaehlerian manifold, Y. Kubo [6] proved

THEOREM B. *Let M be a real n-dimensional Bochner-Kaehlerian manifold with constant scalar curvature R. If the length of the Ricci tensor is not greater than $R/\sqrt{n-2}$, $n \geqq 4$, then M is a space of constant holomorphic sectional curvature.*

Note that the square of the length of the Ricci tensor is greater than or equal to R^2/n, so the Ricci tensor has been "pinched". The inequality in Theorem A is the best possible. But the inequality in Theorem B is less than perfect. So we improve the inequality in Theorem B and obtain the following theorems.

THEOREM 1. *Let M be a real $n(>4)$-dimensional Bochner-Kaehlerian manifold with constant scalar curvature R. If the square of the length of the Ricci tensor is less than $(n^3-2n^2+32)R^2/(n-4)^2(n+2)^2$, then M is of constant holomorphic sectional curvature.*

THEOREM 2. *Let M be a real 4-dimensional Bochner-Kaehlerian manifold. If the scalar curvature is non-zero constant, then M is of constant holomorphic sectional curvature.*

§ 2. Bochner-Kaehlerian manifolds

Let M be a real $n(\geqq 4)$-dimensional Kaehlerian manifold. Then the Riemannian metric g_{ij} and the almost complex structure $J_i{}^h$ satisfy the following equations:

$$J_i{}^a J_a{}^h=-\delta_i^h, \quad g_{ab}J_i{}^a J_j{}^b=g_{ij}, \quad \nabla_k J_i{}^h=0, \quad \nabla_k g_{ij}=0, \tag{2.1}$$

where ∇_k denotes the operator of covariant differentiation with respect to g_{ij}.

Let $R_{hij}{}^k$ be the Riemannian curvature tensor and put $R_{ij}:=R_{aij}{}^a$ (Ricci tensor), $R:=g^{ab}R_{ab}$ (scalar curvature) and $H_{ij}:=J_i{}^a R_{aj}$.

We define the Bochner curvature tensor of M [8] by

$$\begin{aligned}B_{hijk}:=R_{hijk}-\frac{1}{n+4}(R_{ij}g_{hk}-R_{ik}g_{hj}+g_{ij}R_{hk}-g_{ik}R_{hj}+H_{ij}J_{hk}-H_{ik}J_{hj}+J_{ij}H_{hk}\\ -J_{ik}H_{hj}-2H_{hi}J_{jk}-2J_{hi}H_{jk})+\frac{R}{(n+2)(n+4)}(g_{ij}g_{hk}-g_{ik}g_{hj}+J_{ij}J_{hk}-J_{ik}J_{hj}\\ -2J_{hi}J_{jk}).\end{aligned} \tag{2.2}$$

M is called a Bochner-Kaehlerian manifold if the Bochner curvature tensor vanishes.

By the straightforward computation, we have

LEMMA 1 [8]. *If a Bochner-Kaehlerian manifold M is an Einstein one, then M is of constant holomorphic sectional curvature.*

On the other hand, Y. Kubo [6] proved

LEMMA 2. *If a Bochner-Kaehlerian manifold M has the constant scalar curvature, then we have*

$$n(n+2)R_a{}^b R_b{}^c R_c{}^a-2(n+1)RR_{ab}R^{ab}+R^3=0. \tag{2.3}$$

§ 3. Proofs of theorems

We define the Einstein tensor S_{ij} by

$$S_{ij}:=R_{ij}-\frac{R}{n}g_{ij}.$$

Then we have $S_i{}^a J_a{}^j=J_i{}^a S_a{}^j$ since $R_i{}^a J_a{}^j=J_i{}^a R_a{}^j$. Moreover we see that

$$\text{trace } S:=S_a{}^a=0, \tag{3.1}$$

$$\text{trace } S^2:=S_{ab}S^{ab}=R_{ab}R^{ab}-\frac{R^2}{n}\geqq 0, \tag{3.2}$$

$$\text{trace } S^3:=S_a{}^b S_b{}^c S_c{}^a=R_a{}^b R_b{}^c R_c{}^a-\frac{3}{n}RS_{ab}S^{ab}-\frac{1}{n^2}R^3. \tag{3.3}$$

M is an Einstein space if and only if *trace* S^2 vanishes.

Substituting (3.2) and (3.3) in (2.3), we have

$$(3.4)\qquad n(n+2)\,trace\ S^3+(n+4)R\ trace\ S^2=0.$$

M. Okumura [7] proved

LEMMA 3. *Let* c_i, $i=1,2,\cdots,m$, *be real numbers satisfying*

$$\sum_{i=1}^{m} c_i=0 \text{ and } \sum_{i=1}^{m} c_i^2=k^2 \quad (k\geqq 0).$$

Then we have

$$-\frac{m-2}{\sqrt{m(m-1)}}\,k^3\leqq\sum_{i=1}^{m} c_i^3\leqq\frac{m-2}{\sqrt{m(m-1)}}\,k^3.$$

We put $f^2:=trace\ S^2 (f\geqq 0)$. From the commutativity of $S_i{}^j$ and $J_i{}^j$, we see that every characteristic root of $S_i{}^j$ is multiple one. Combining this fact with Lemma 3, we have

$$(3.5)\qquad -\frac{n-4}{\sqrt{2n(n-2)}}f^3\leqq trace\ S^3\leqq\frac{n-4}{\sqrt{2n(n-2)}}f^3.$$

Applying the above inequality, (3. 4) yields the following inequality:

$$(3.6)\qquad f^2\left\{\frac{n+4}{n(n+2)}R-\frac{n-4}{\sqrt{2n(n-2)}}f\right\}\leqq 0\leqq f^2\left\{\frac{n+4}{n(n+2)}R+\frac{n-4}{\sqrt{2n(n-2)}}f\right\}.$$

Under the assumption of Theorem 1, we see that $f^2=0$, that is, M is a Einstein space. Therefore, from Lemma 1, Theorem 1 has been proved.

If dim M is 4, we see that *trace* $S^3=0$ since the characteristic roots of $S_i{}^j$ are $\kappa, \kappa, -\kappa$ and $-\kappa$. From (3. 4), we have

$$R\ trace\ S^2=0.$$

Therefore, if R is non-zero constant, then M is an Einstein space, whence M is of constant holomorphic sectional curvature. Thus Theorem 2 has been proved.

§4. Exaples of the Bochner-Kaehlerian manifolds

Let M_1 be a real $(n-2)$-dimensional $(n\geqq 4)$ Kaehlerian manifold of constant holomorphic sectional curvature $c(\neq 0)$ and M_2 be a real 2-dimensional Kaehlerian manifold of constant holomorphic sectional curvature $-c$. The product manifold $M=M_1\times M_2$ is a typical example of Bochner-Kaehlerian manifold with constant scalar curvature which is not of constant holomorphic sectional curvature. Moreover we have

$$(4.1)\qquad R=\frac{(n-4)(n+2)}{4}c$$

and

$$(4.2)\qquad R_{ab}R^{ab}=\frac{n^3-2n^2+32}{16}c^2.$$

Therfore, if $n > 4$, we obtain

$$(4.3) \qquad R_{ab}R^{ab} = \frac{n^3 - 2n^2 + 32}{(n-4)^2(n+2)^2} R^2.$$

This shows that the inequality in Theorem 1 is the best possible.

If $n=4$, scalar curvature of M is zero constant. This shows that there exists a real 4-dimensional Bochner-Kaehlerian manifold with constant scalar curvature 0 which is not of cnostant holomorphic sectional curvature.

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