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The WKB Method and Energy Levels of $c\bar{c}$ Bound States in the Standard Potential

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平野雅宣・山口　宏：WKBの方法とスタンダードポテンシャルに束縛されている$c\bar{c}$のエネルギー準位

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Abstract

We offer a solution of the Schroedinger equation for the charmed quark-antiquark ($c\bar{c}$) bounded in the standard potential; $V(r)=-\alpha_g/r+\lambda r$, by the method of Wentzel-Kramers-Brillouin (WKB).

L=0 states (S-wave radially excited states) are investigated. We determine the parameters proper to this system, inputting the mass splittings of charmonium ($c\bar{c}$). The results are very striking and not less than the computer analyses.

§I Introduction

The spectroscopy of new mesons (charmonium and upsilon) has been investigated vigorously by assuming the potential between heavy quark and antiquark ($c\bar{c}$, $b\bar{b}$). For these mesons, the nonrelativistic treatment is regarded as a good approximation.

With this consideration, three types of the model potential are proposed at present. Two of them are simple ones and the other is the sophisticated one.

The first potential[1),2)] is

$$V(r)=-\frac{\alpha_g}{r}+\lambda r, \tag{1}$$

which we name the standard potential. The first term is motivated by the asymptotic freedom at a short distance besed on Q. C. D.. The second term is motivated by the quark confinment at a long distance which is suggested by the lattice guage theory, the dual string model

and the mass level ordering (Ψ-series) due to the phenomenological study.

Another one[3] is constructed by Fourier transformation of the gluon exchage like amplitude, which is

$$V(q^2)=-\frac{4}{3}\frac{1}{q^2}\frac{\alpha_s(\Lambda^2)}{1+\frac{33-2N}{12\pi}\alpha_s(\Lambda^2)\,ln(1+\frac{q^2}{\Lambda^2})} \quad . \tag{2}$$

This is interpolated to the infra-red regions beyond the discontinuity around $q^2\leqq\Lambda^2$ from the dressed single gluon exchange amplitude, which behaves as$\sim 1/q^2(1+const.\ ln\ q^2/\Lambda^2)$.

Of course, the admixture of (1) and (2) were attempted recently.[4]

The other is given as follows in more sophisticated fashions[5),6)],

$$\begin{aligned} V(r)&=-\frac{\alpha_g}{r} && r\leq r_1,\\ &=\log\frac{r}{r_o} && r_1\leq r\leq r_2,\\ &=\lambda\, r && r\geq r_2. \end{aligned} \tag{3}$$

where r_1 and r_2 are determined by continuing $V(r)$ and its first derivative.

The logarithmic potential at intermediate distances (around 0.5 to 1.5 fm) is a new proposal, where no theoretical dogma exists. The potential was suggested to explan the $\Upsilon'-\Upsilon$ mass splitting identical to the $\Psi'-\Psi$ mass splitting[7].

We selected the first type of potential due to the simplicity. We have investigated the mass spectroscopy of $c\bar{c}$ on the basis of the above potential[8]. We regarded the linear potential as the main interaction and treated the Coulombic potential perturbatively. For the analytical calculation, we introduced the trial harmonic wave function and determined its variable parameter by the variation method. We obtained the averaged mass levels and the spin-dependent mass splittings of charmonium by the above wave function[9].

However, the Coulombic potential is singular at r=0. The perturbative treatment of this is rough for the determination of the energy levels.

We will therefore try to solve the Schroedinger equation, considering both the linear and the Coulombic potential. We cannot solve the wave equation analytically, but we can compute it numerically or calculate it by the more reliable approximation method. The former is at present being developed. The latter is the WKB quantization method[10] which we will report in this paper.

The WKB method is a good approximation for the Coulomb potential singular at r=0. The obtained energy levels are the same as the exact ones. We therefore apply the WKB method to solving the Schroedinger equation of $c\bar{c}$ bounded in the standard potential in **§ II.**

In **§ III**, we will complete the potential model, determining the model parameters of the $c\bar{c}$ system; the reduced quark mass (μ), the strength of the Coulombic potential (α_g) and that of the linear potential (λ).

§ II The WKB approximation and the L=0 spectroscopy

Based on the theoretical and phenomenological speculations, we consider the potential

$$V(r)=-\frac{\alpha_g}{r}+\lambda r. \tag{4}$$

We write the Shroedinger equation for quark and antiquark in the potential as

$$-\frac{1}{2\mu}\frac{d^2}{dr^2}+(-\frac{\alpha_g}{r}+\lambda r+\frac{L(L+1)}{2\mu r^2}-E)R(r)=0, \tag{5}$$

where $R(r)$ is the radial wave function ($r\Psi(r)$) and μ is the reduced mass. It is convenient to make a change of variables,

$$x=\frac{r}{l}, \tag{6}$$

where $l=(2\mu\lambda)^{-\frac{1}{3}}$

The equation then reads

$$\left\{\frac{d^2}{dx^2}-(\frac{c}{x}+x+\frac{L(L+1)}{x^2}-\frac{E}{\varepsilon})\right\}R(x)=0, \tag{7}$$

where we have introduced two parameters: $c=\alpha_g 2\mu l$ and $\varepsilon=(2\mu l^2)^{-1}$.

We note that the eq. (7) is characterized by only the above two parameters in spite of the arbitrary set of μ α_g and λ. The dimensionless parameter c means the strength of the Coulombic interaction.

We can solve the wave equation numerically and obtain the energy eigen values. Inputting the mass differences of the charmonium, we determin the c and $N=E/\varepsilon$. The result of the computer anallyses will be reported in the near future.

However. a very good approximate solution to the stationary-state eigenvalue problem is given by the WKB method[10]. In this paper, we solve the equation by this method.

For S states ($L=0$), the WKB quantization appropriate to eq.(7) gives

$$\int_o^{x_1}\sqrt{g_o(x)}\,dx=(n+\frac{3}{4})\pi, \tag{8}$$

where

$$g_o(x)=\frac{E}{\varepsilon}-x+\frac{c}{x}$$

$$=N-x+\frac{c}{x}, \tag{9}$$

and n is the number of nodal points of the wave functions. $x_1(>0)$ is the first classical turning point, which is the solution of $g_o(x)=0$.

For $L\neq 0$ states, the WKB qauntization appropriate to eq. (7) is

$$\int_{x_0}^{x_1}\sqrt{g_l(x)}\,dx=(n+\frac{1}{2})\pi, \tag{10}$$

where

$$g_l(x)=N-x+\frac{c}{x}-\frac{(L+1/2)^2}{x^2}, \tag{11}$$

with the replacement of $L(L+1)\rightarrow(L+1/2)^2$.

Equation (8) and (10) cannot be integrated analytically. However, the integration can be represented in terms of sums of complete elliptic integrals of the first, second and third kind[11]. In the following, the integration of eq. (8) is explained in detail. With respect to the integration of eq. (10), we will publish the results in our next paper.

Eqaution (8) is rewritten as

$$\int_o^{x_1}\sqrt{g_o(x)}\,dx=\int_o^{x_1}\frac{-x^2+Nx+c}{\sqrt{-x^3+Nx^2+cx}}dx. \tag{12}$$

Making a change of variable $x=\nu^2(1-u^2)$, eq. (12) is as follows,

$$\begin{aligned}\int_o^{x_1}\sqrt{g_o(x)}\,dx=&\frac{-2\nu^4}{\sqrt{\nu^2+\mu^2}}\int_0^1\frac{u^4}{\sqrt{(1-u^2)(1-k^2u^2)}}du\\&+\frac{4\nu^4-2N\nu^2}{\sqrt{\nu^2+\mu^2}}\int_0^1\frac{u^2}{\sqrt{(1-u^2)(1-k^2u^2)}}du\\&+\frac{2c+2\nu^2N-2\nu^4}{\sqrt{\nu^2+\mu^2}}\int_0^1\frac{1}{\sqrt{(1-u^2)(1-k^2u^2)}}du\end{aligned} \tag{13}$$

where

$$\nu^2=\frac{1}{2}\{\sqrt{N^2+4c}+N\},$$

$$\mu^2=\frac{1}{2}\{\sqrt{N^2+4c}-N\},$$

$$k^2=\frac{\nu^2}{\nu^2+\mu^2}$$

and $(\nu^2,-\mu^2)$ is the solution of $Nx-x^2+c=0$.

For brevity, we replace the first, second and third integration terms of eq. (13) with I_4, I_2, and I_0 respectively,

$$I_4=\int_0^1\frac{u^4}{\sqrt{(1-u^2)(1-k^2u^2)}}du,$$

$$I_2=\int_0^1\frac{u^2}{\sqrt{(1-u^2)(1-k^2u^2)}}du,$$

$$I_0=\int_0^1\frac{1}{\sqrt{(1-u^2)(1-k^2u^2)}}du, \tag{14}$$

Using the recurrence formula of the above integrals;

$$\begin{aligned}&(2m+4)a_0I[m+3]+(2m+3)a_1I[m+2]+(2m+2)a_2I[m+1]+(2m+1)a_3I[m]+2ma_4\\&I[m-1]=2x^m\sqrt{\varphi(x)},\end{aligned} \tag{15}$$

where

$$I[m]=\int\frac{x^m}{\sqrt{\varphi(x)}}dx,$$

$$\varphi(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4, \tag{16}$$

and cosidering $a_1=a_3=0$ in this case, we obtain the following formula

$$I_4=\frac{2(1+k^2)}{3k^2}I_2-\frac{1}{3k^2}I_0. \tag{17}$$

We also rearrange 1_2 as

$$I_2=\frac{1}{k^2}I_0-\frac{1}{k^2}\int_0^1\sqrt{\frac{1-k^2u^2}{1-u^2}}\,du \tag{18}$$

We replace I_0 and the second term of eq. (18) with $F(k)$ and $E(k)$, which are the complete elliptic integrals of the first kind and the second kind respectively;

$$F(k)=I_0=\int_0^1\frac{1}{\sqrt{(1-u^2)(1-k^2u^2)}}\,du$$
$$E(k)=\int_0^1\sqrt{\frac{1-k^2u^2}{1-u^2}}\,du \tag{19}$$

Substituting eqs. (17), (18) and (19) into (13), we obtain the final expression of the integrals

$$\int_0^{x1}\sqrt{g_0(x)}\,dx=\sqrt{A}\left(\frac{A-N}{3}F(k)+\frac{2N}{3}E(k)\right), \tag{20}$$

where

$$A=\nu^2+\mu^2\ =\sqrt{N^2+4c}\ , \tag{21}$$

$$k^2=\frac{\nu^2}{\nu^2+\mu^2}=\frac{\sqrt{N^2+4c}+N}{2\sqrt{N^2+4c}} \tag{22}$$

Therefore the WKB quatization condition for S-state becomes

$$\sqrt{A}\left(\frac{A-N}{3}F(k)+\frac{2N}{3}E(k)\right)=(n+\frac{3}{4})\pi. \tag{23}$$

§ III Model Parameters for Charmonium Spectra

We will construct the model for quark-antiquark bound states by determining the parameters; μ, α_g and λ. As we mentioned in the previous section, energy levels are given by only two parameters ($N$and c). Therefore experimental information of charmonium mass spectroscopy (for example, mass splittings) is not sufficient for the determination of μ, α_g and λ. If we try to select the three parameters, we need the other independent experimental inputs, which are the leptonic decay widths of charmonium and so on.

For the present, we investigate the model parameters by examining the mass levels of charmonium. We can find sets of N and c from the condition of eqs. (21), (22) and (23). These are plotted in Fig. 1 for $1S(n=0), 2S(n=1), 3S(n=2)$ and $4S(n=3)$ states, refering to the table of $F(k)$ and $E(k)$[12]. The mass difference between mS and $1S$ is given by

$$\Delta E_{m1}=E(mS)-E(1S)=\{N(n=m-1)-N(n=0)\}\varepsilon$$
$$=\{N(n=m-1)-N(n=0)\}\left(\frac{\lambda^2}{2\mu}\right)^{1/3} \tag{24}$$

The mass difference ΔE_{m1} versus c is plotted in Fig. 2 in unit of ε.

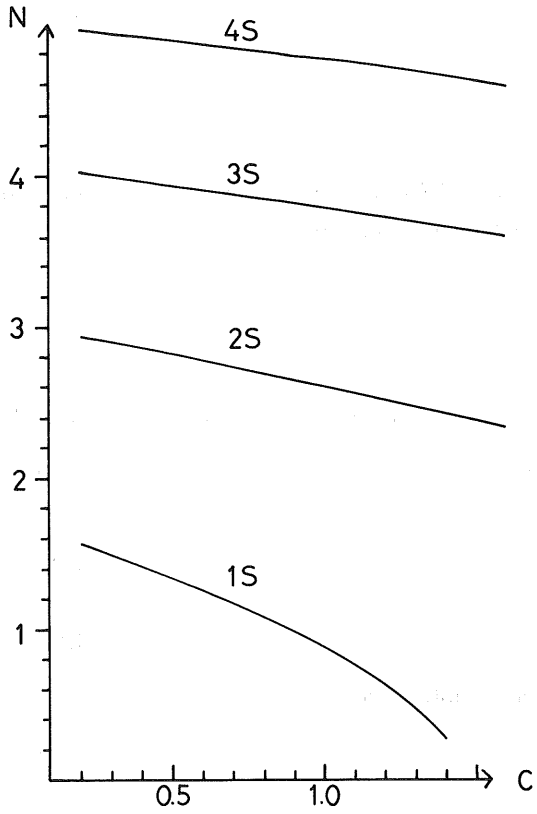

Fig. 1. N versus c is plotted for 1S, 2S, 3S and 4S states. N means the levle energy in unit of ε. c is the strenght of the Coulombic potential.

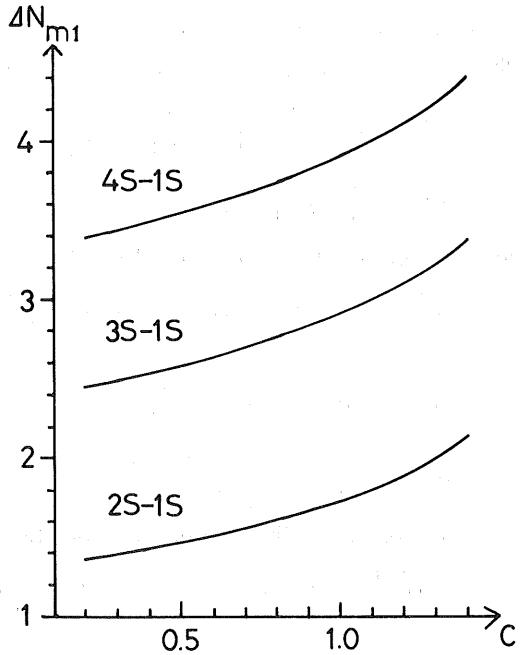

Fig. 2. The energy difference in unit of ε; $\Delta E_{m1}/\varepsilon = \Delta N_{m1} = N(n=m-1) - N(n=0)$ is plotted for $m=2, 3$ and 4 .

If we obtain one of three parameters (μ, α_g, λ), we determine the others, putting mass splitting values into ΔE_{m1}. So we will assume the value of λ, which is the flavor independence suggestd by theoretical arguments and phenomenological analyses. This parameter is also related to the Regge slope α'

$$\lambda = \frac{1}{2\pi\alpha'} , \tag{25}$$

beging provided by a field-theoeretical interpretation of the string picture of hadrons[13]. Equation (25) suggests $\lambda = 0.16$ Gev^2 if we take $\alpha' = 1$ Gev^{-2}. Practically we investigate two cases; $\lambda = 0.171$ Gev^2 and $\lambda = 0.183$ Gev^2 which are determined by DESY group[5] and CORNELL group[2] respectively.

For these λ's, we obtain the reduced quark mass (μ) with values of c, substituting experimental mass splittings of charmonium into eq. (24). Then we get the relation of α_g versus μ from $c = \alpha_g\, 2\mu l$. These results are shown in Fig. 3, (a) and (b).

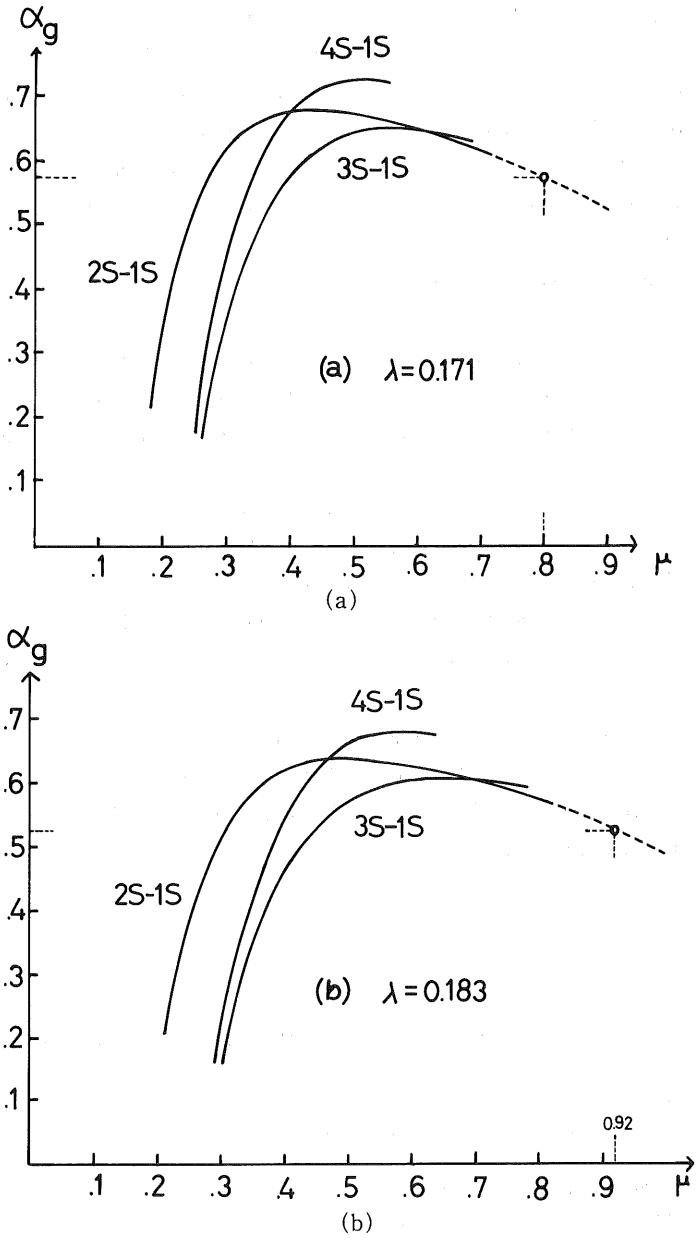

Fig. 3 The relation of α_g and the reduced quark mass (μ) is given, imposing the condition to ΔE_{m_1} = the experimental mass splitting of charmonium in Fig. 2. The mass of 3S and 4S states are not established completely
(a) The case of $\lambda=0.171$ (DESY parameter).
(b) The case of $\lambda=0.183$ (CORNELL parameter).
The broken line is drawn by hand. $\mu=0.8$ or $\mu=0.92$ is explained in the paper.

We note that the graphs in both Fig. 1/Fig. 2 becomes broken at $c \approx 1.5$. When we neglect the Coulombic term ($c=0$) in the potential, all energy levels take the plus sign. On the other hand, the Coulombic term only gives the minus values of energy levels for the bound state as well known. Therefore the stronger the Coulombic term becomes, the lower do the energy levels. The energy of 1S state changes from the plus value to the minus one near at $c \approx 1.5$. Our WKB treatment to solve the Schroedinger equation is limited to the plus energy solution ($E>0$). We must modify the calculation in § II in the case of minus energy. We postpone a solution of this problem and draw the solution by hand to large c values, shown by the dotted line in Fig. 3, (a) and (b).

Investigating whether our WKB method is good or not, we compare our parameter sets with those of DESY and CORNELL which are determined by computer analyses. For these purposes, we select α_g values from Fig. 3, (a) and (b), inputting $\mu=0.80$ Gev, $\lambda=0.171$ Gev^2 (DESY parameters) and $\mu=0.92$ Gev, $\lambda=0.183$ Gev^2 (CORNELL parameters). We obtain results: $\alpha_g=0.575$ for the former parameters and $\alpha_g=0.525$ for the latter parameters. These values are very close to $\alpha_g=0.55$ (DESY) and $\alpha_g=0.52$ (CORNELL)[2),5)].

This means that our WKB method is appropriate to solve the Shroedinger equation including the singular potential at the origin($r=0$).Certainly the above method is better than the variation method developed in the previous paper[8)] and not less than the numerical analyses by use of the computer.

Problems to be solved are these:(1) calculating the energy levels in the case of the minus sign by the WKB quantization method, (2) $L \neq 0$ spectra of charmonium and (3) the mass spectroscopy of the upsilon ($b\bar{b}$ boud etates) in the same method. The analyses for the problem will be published in the next. paper.

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