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# On the Noncentral $\chi^2$ distribution K(n, $\lambda$) and its Moments

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猪野富秋：非心$\chi^2$分布 K(n, $\lambda$)とその積率について

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**Abstract**

Usually we get a probability density function of the noncentral distribution K(n, $\lambda$), where n is the degrees of freedom $\lambda$ is the noncentral parameter, by the .invertion of its characteristic function.

It is too complicated to discover its moments directly.

In this paper, we give first a method of constructing the probability density function of K(n, $\lambda$), considering that the degrees of freedom n is essentially one.

Next, by the aid of Laguarre's polynominals, we describe a systematic procedure to arrive at its moments.

Finally as a result of our procedure, we give some account of its cumulants.

## § 1. Abbrevations and Notations

In this paper, the following abbrevations and notatiohs are used.

(1) Probability density function : p.d.f.

(2) Degrees of freedom : d.f.

(3) A random variable X has a normal distribution with mean m and variance $\sigma^2$ : X$\sim$N (m, $\sigma^2$), and its p.d.f. : N(x|m, $\sigma^2$)

(4) A random variable X has a $\chi^2$ distribution with degrees of freedom n : X$\sim \chi_n^2$ and its p.d.f. : k(x|n)

(5) A random variable X has a noncentral $\chi^2$ distribution with degrees of freedom n and a noncentral parameter $\lambda$ : X$\sim$K(n, $\lambda$) , and its p.d.f. : K(x|n, $\lambda$)

(6) p-dimentional random vector $\underline{X}$ has a p-dimentional normal distribution with mean

vector $\underline{m}$ and a variance covariance matrix $\Sigma$: $\underline{X} \sim N_p(\underline{m}, \Sigma)$, and its p.d.f.: $N(\underline{x}|\underline{m}, \Sigma)$

(7) Transpose of A : A'

(8) Jacobian of transformation from V to W : $J(V \rightarrow W)$

(9) Two random variables X and Y are independent : $X \perp Y$

## § 2. K(l, m)

When $X \sim N(m, l^2)$, viz.

$$N(x|m, l^2) = (2\pi)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(x-m)^2\right] \quad (2.1)$$

if we put $Y = X^2$, then from (2. 1) its p.d.f. f(y) is given as follows.

$$\begin{aligned} f(y) &= N(x|m, l^2)\left|\frac{dx}{dy}\right| \\ &= (2\pi)^{-\frac{1}{2}}\left[\exp\left[-\frac{1}{2}(-\sqrt{y}-m)^2\right] + \exp\left[-\frac{1}{2}(\sqrt{y}-m)^2\right]\right]\frac{1}{2\sqrt{y}} \\ &= (2\pi)^{-\frac{1}{2}}\frac{1}{2\sqrt{y}} \exp\left(-\frac{m^2}{2}\right)\exp\left(-\frac{y}{2}\right)\left[\exp(\sqrt{y}\,m) + \exp(-\sqrt{y}\,m)\right] \\ &= (2\pi)^{-\frac{1}{2}}\frac{1}{\sqrt{y}} \exp\left(-\frac{m^2}{2}\right)\exp\left(-\frac{y}{2}\right)\sum_{k=0}^{\infty}\frac{(\sqrt{y}\,m)^{2k}}{(2k)!} \\ &= (2\pi)^{-\frac{1}{2}} \cdot \exp\left(-\frac{m^2}{2}\right)\sum_{k=0}^{\infty}\frac{m^{2k}}{(2k)!}\, y^{\frac{2k+1}{2}-1} \exp\left(-\frac{y}{2}\right) \end{aligned} \quad (2.2)$$

For the $\Gamma$-function, the following relations are held.

$$\Gamma\left(k+\frac{1}{2}\right) = \sqrt{\pi}\frac{(2k)!}{k!\,2^{2k}} \qquad (k=0, 1, 2, 3, ) \quad (2.3)$$

and the p.d.f. of $\chi^2$ distribution with freedom degrees of n k(x|n) is modified to the following from.

$$x^{\frac{n}{2}-1}\exp\left(-\frac{x}{2}\right) = 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)k(x|n) \quad (2.4)$$

By(2. 2), (2. 3) and (2. 4), we have

$$\begin{aligned} f(y) &= (2\pi)^{-\frac{1}{2}}\exp\left(-\frac{m^2}{2}\right)\sum_{k=0}^{\infty}\frac{m^{2k}}{(2k)!}\,2^{\frac{2k+1}{2}}\,\Gamma\left(\frac{2k+1}{2}\right)k(y|2k+1) \\ &= \exp\left(-\frac{m^2}{2}\right)\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{m^2}{2}\right)^k k(y|2k+1) \end{aligned} \quad (2.5)$$

(2. 5) means

$$Y \sim K(l, m) \quad (2.6)$$

## § 3. K(n, λ)

when $\underline{X} \sim N_n(\underline{m}, I(n))$, viz.

$$N_n(\underline{x}|\underline{m}, I(n)) = (2\pi)^{-\frac{n}{2}} \exp[-\frac{1}{2}(\underline{x}-\underline{m})'(\underline{x}-\underline{m})] \quad (3.1)$$

where $\underline{m}$ and I(n) are as follows.

$$\underline{m} = \begin{pmatrix} m \\ m \\ \vdots \\ m \end{pmatrix} = m\underline{1} \qquad I(n) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Now, we transform the n-dimentional random vector $\underline{X}$ to the n-dimentional random vector $\underline{Y}$ by the orthogonal matrix L, such that

$$\underline{X} = L\underline{Y} \quad (3.2)$$

where

$$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \qquad \underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} \qquad L = \begin{pmatrix} \frac{1}{\sqrt{n}} & \\ \frac{1}{\sqrt{n}} & \triangle \\ \vdots & \\ \frac{1}{\sqrt{n}} & \end{pmatrix} \quad (3.3)$$

From (3. 2) and (3. 3) we have the following elations.

$$\underline{Y} = L'\underline{X} \quad (3.4)$$

$$Y_1 = \frac{1}{n}(X_1 + X_2 + \cdots + X_n) = \sqrt{n}\,\overline{X} \quad (3.5)$$

$$(\underline{X}-\underline{m})'(\underline{X}-\underline{m}) = (L\underline{Y}-\underline{m})'(L\underline{Y}-\underline{m}) = (\underline{Y}'L'-\underline{m}')(L\underline{Y}-\underline{m})$$

$$= (Y_1 - \sqrt{n}\,m)^2 + Y_2^2 + Y_3^2 + \cdots + Y_n^2) \quad (3.6)$$

$$\mathrm{mod}\ [J(\underline{X} \to \underline{Y})] = 1 \quad (3.7)$$

By (3.1)~(3.7), we get the p.d.f. f(y) of Y as follows.

$$f(\underline{y}) = (2\pi)^{-\frac{1}{2}} \exp[-\frac{1}{2}(y_1 - \sqrt{n}\,m)^2 (2\pi)^{-\frac{n-1}{2}} \exp[-\frac{1}{2}(y_2^2 + y_3^2 + \cdots + y_n^2)] \quad (3.8)$$

as L is an orthogonal matrix, we have

$$\underline{X}'\underline{X} = \underline{Y}'\underline{Y} = Y_1^2 + Y_2^2 + \cdots + Y_n^2 \quad (3.9)$$

From (2. 6) and (3.8), we find the following properties.

$$Y_1 \perp Y_2^2 + Y_3^2 + \cdots + Y_n^2 \quad (3.10)$$

$$Y_1 \sim K(1, \sqrt{n}\,m) \qquad (3.11)$$

$$Y_2^2 + Y_3^2 + \cdots + Y_n^2 \sim \chi_{n-1}^2 \qquad (3.12)$$

(3. 10), (3. 11), (3. 12) and the reproductive property for the degree of freedoms show that the p.d.f. f(x) of $X = \underline{X}'\underline{X}$ has the following form.

$$f(x) = \exp\left[-\frac{1}{2}(\sqrt{n}\,m)^2\right] \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{nm^2}{2}\right)^k k(x|n+2k) \qquad (3.13)$$

(3. 13) is the p.d.f. of $K(n, \sqrt{n}\,m)$.
Generally the p.d.f. of $K(n, \lambda)$ is given as follows.

$$K(x|n, \lambda) = \exp\left(-\frac{\lambda^2}{2}\right) \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\lambda^2}{2}\right)^k k(x|n+2k) \qquad (\lambda > 0)$$

where $\lambda$ is called the noncentral parameter; in (3. 13) tnis noncentral parameter $\sqrt{n}\,m$ means the length of a vector $\underline{m}$.

## § 4. The Moments of K(n, λ)

When $X \sim K(n, \lambda)$, the following properties are well known.

1. The moment generating function g(t) of X is given as follows.

$$g(t) = (1-2t)^{-\frac{n}{2}} \exp\left(\frac{\lambda t}{1-2t}\right) \qquad (4.1)$$

2. Its cumulants of order r are given as follows.

$$\kappa_r = 2^{r-1}(r-1)!(n+\lambda r) \qquad (r = 1, 2, 3, \cdots) \qquad (4.2)$$

We try to get a concrete description of its r-th moment $\alpha_r$ by Laguerre's polynominals. Laguerre's polynominals $L_n(x)$ are given by the following equalities.

$$L_n(x) = \exp(x) \frac{d^n}{dx^n} [x^n \exp(-x)] \qquad (4.3)$$

(4. 3) gives the following expression.

$$L_n(x) = (-1)^n \left[x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2(n-1)^2}{2!} x^{n-2} - \cdots + (-1)^n n!\right] \qquad (4.4)$$

Hence, we have

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= -x+1 \\ L_2(x) &= x^2-4x+2 \\ L_3(x) &= -x^3+9x^2-18x+6 \\ L_4(x) &= x^4-16x^3+72x^2-96x+24 \\ &\cdots\cdots \end{aligned} \qquad (4.5)$$

On the other hand, by the generating function of $L_n(x)$, the following relation is held.

$$\frac{1}{1-t} \exp\left(\frac{-t\lambda}{1-t}\right) = \sum_{r=0}^{\infty} L_r(\lambda) \frac{t^r}{r!} \qquad (4.6)$$

in (4. 6) if we put t=2t and $\lambda=-\frac{\lambda}{2}$, then we have

$$\frac{1}{1-2t}\exp\left[\frac{-2t}{1-2t}\left(-\frac{\lambda}{2}\right)\right]=\sum_{r=0}^{\infty}L_r\left(-\frac{\lambda}{2}\right)2^r\frac{t^r}{r!} \tag{4.7}$$

The first member of (4. 7) expresses the moment generating function of K(2, $\lambda$), and the r-th moments of K(2, $\lambda$) (we write them $\beta_r$) are obtained as follows.

$$\beta_r=2^r L_r\left(-\frac{\lambda}{2}\right) \qquad (r=0,1,2,3,\cdots) \tag{4.8}$$

therefore

$$\beta_r=2^r(-1)^r\left[\left(-\frac{\lambda}{2}\right)^r-\frac{r^2}{1!}\left(-\frac{\lambda}{2}\right)^{r-1}+\frac{r^2(r-1)^2}{2!}\left(-\frac{\lambda}{2}\right)^{r-2}-\cdots+(-1)^r r!\right] \tag{4.9}$$

the details are given as follows.

$$\begin{aligned}
&\beta_0=1\\
&\beta_1=\lambda+2\\
&\beta_2=\lambda^2+8\lambda+8\\
&\beta_3=\lambda^3+18\lambda^2+72\lambda+48\\
&\beta_4=\lambda^4+32\lambda^3+288\lambda^2+768\lambda+384\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{4.10}$$

Now, we transform the moment generating function g(t) of K(n, $\lambda$) into the following form.

$$\begin{aligned}
g(t)&=(1-2t)^{-\frac{n}{2}}\exp\left(\frac{\lambda t}{1-2t}\right)\\
&=(1-2t)^{-\frac{n}{2}+1}(1-2t)^{-1}\exp\left[\frac{-2t}{1-2t}\left(-\frac{\lambda}{2}\right)\right]\\
&=(1-2t)^{-\frac{n-2}{2}}\sum_{k=0}^{\infty}\frac{\beta_k}{k!}t^k
\end{aligned} \tag{4.11}$$

In this place, we introduce the following notations:

$$\begin{aligned}
&p_0=1\\
&p_1=n-2\\
&p_2=(n-2)n\\
&p_3=(n-2)n(n+2)\\
&\cdots\cdots\cdots\cdots\\
&p_k=(n-2)n(n+2)\cdots(n+2k-4)
\end{aligned}$$

So that, we have

$$\begin{aligned}
g(t)&=\left(\sum_{j=0}^{\infty}\frac{p_j}{j!}t^j\right)\left(\sum_{k=0}^{\infty}\frac{\beta_k}{k!}t^k\right)\\
&=\left(1+\frac{p_1}{1!}t+\frac{p_2}{2!}t^2+\cdots+\frac{p_j}{j!}t^j+\cdots\right)\left(1+\frac{\beta_1}{1!}t+\frac{\beta_2}{2!}t^2+\cdots+\frac{\beta_k}{k!}t^k+\cdots\right)\\
&=1+(p_1+\beta_1)\frac{t}{1!}+\left(p_2+\frac{2!}{1!1!}p_1\beta_1+\beta_2\right)\frac{t^2}{2!}+\cdots\\
&\quad+(p_r++{}_rC_1p_{r-1}\beta_1+{}_rC_2p_{r-2}\beta_2+\cdots+\beta_r)\frac{t^r}{r!}-\cdots
\end{aligned} \tag{4.12}$$

Hence, we have following result.

**Theorem :** When $X \sim K(n,\lambda)$, the r-th moments $\alpha_r$ of X have the following expression.

$$\alpha_r = \sum_{j=0}^{r} {}_rC_j p_{r-j} \beta_j \qquad (r=0, 1, 2, 3 \cdots)$$

where

$$p_0 = 1$$
$$p_k = (n-2)n(n+2)\cdots(n+2k-4) \qquad (k=1, 2, 3, \cdots)$$
$$\beta_j = 2^j L_j\left(-\frac{\lambda}{2}\right) \qquad (j=0, 1, 2, 3, \cdots)$$

## § 5. Note on Cumulants

When $X \sim K(n, \lambda)$, for some of its moments of lower order, we have

$$\begin{aligned} \alpha_1 &= n+\lambda \\ \alpha_2 &= (n+\lambda)^2 + 2(n+2\lambda) \\ \alpha_3 &= (n+\lambda)^3 + 6(n+\lambda)(n+2\lambda) + 8(n+3\lambda) \\ \alpha_4 &= (n+\lambda)^4 + 12(n+\lambda)(n+2\lambda) + 44n^2 + 176n\lambda + 144\lambda^2 + 48(n+4\lambda) \\ &\cdots\cdots \end{aligned} \qquad (5.1)$$

From (4. 2) and (5. 1), we can suppose that $\kappa_r$ will appear in the terms of $\alpha_r$ and their order with respect to n and $\lambda$ is one.

In order to verify this assumption, we make the following table.

| terms of $\alpha_r$ | coefficients of n | coefficients of $\lambda$ |
|---|---|---|
| ${}_rC_0 p_r \beta_0$ | $-\frac{(r-2)!}{r!} 2^{r-1} r!$ | |
| ${}_rC_1 p_{r-1} \beta_1$ | $-\frac{(r-3)!}{(r-1)!} 2^{r-1} r!$ | |
| ………… | ………… | |
| ${}_rC_{r-2} p_2 \beta_{r-2}$ | $-\frac{0!}{2!} 2^{r-1} r!$ | |
| ${}_rC_{r-1} p_1 \beta_{r-1}$ | $\frac{1}{1!} 2^{r-1} r!$ | $-(r-1)r!2^{r-1}$ |
| ${}_rC_r p_0 \beta_r$ | | $r \quad r!2^{r-1}$ |
| total | $s_1$ | $s_2$ |

For $s_1$ and $s_2$, we have

$$s_1 = r!2^{r-1}\left[1 - \frac{0!}{2!} - \frac{1!}{3!} - \frac{2!}{4!} - \cdots - \frac{(r-2)!}{r!}\right]$$

$$=r!2^{r-1}[1-(\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{(r-1)r}]$$

$$=r!2^{r-1}[1-[1-(1-\frac{1}{r})]$$

$$=(r-1)!2^{r-1} \tag{5.2}$$

$$s_2=r!2^{r-1}[-(r-1)+r]$$

$$=r!2^{r-1} \tag{5.3}$$

From (5. 2) and (5. 3), we obtain

$$\begin{aligned} s_1n+s_2\lambda &= (r-1)!2^{r-1}n+r!2^{r-1}\lambda \\ &= (r-1)!2^{r-1}(n+\lambda r) \\ &= \kappa_r \end{aligned}$$

Thus, our assumption is verified.

## References


[1] Patnaik, P.B.(1949), The non-central $\chi^2$-and F-distributions, and their applications. Biometrica. Vol.36, p. 202-232.

[2] Kshirsagar, A.M.(1972), Multivariate Analysis. Marcal Dekker, Inc. New York.

[3] Sueoks, S. (1957), Theory of Infinite Series and Orthogonal Functions. Corona Publishing Co. Tokyo.