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Some Results for Weakly Unitarily Invariant Norm

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Abstract

Let M_n be the space of all n x n complex matrices.

A norm $\|\cdot\|$ on M_n is called an operator matrix norm subordinate to the vector norm $n(\cdot)$ on $\mathcal{C}^n$ if $\|\cdot\|$ in defined by $\|A\| = \sup_{x \neq 0} \frac{n(AX)}{n(X)}$ for all $A \in M_n$, and $\|\cdot\|$ is said to be weakly unitarily invariant if for any unitary matrix U, $\|U^*AU\| = \|A\|$ for all $A \in M_n$.

The first result noted in this paper is that the operator norm $\|\cdot\|$ which is weakly unitarily invariant for all rank one matrices is the spectral norm.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be in M_n. Schur product of A and B is defined and denoted by

$$A \circ B = (a_{ij} b_{ij}).$$

Recently, several inequalities for the Schur product have been proved. (c. f. [1], [3], [5] and [6])

We will take Ong's findings [6] in particular and will give a simple proof of his result, with some applications.

1. Weakly Unitarily Invariant Norm.

A map $\|\cdot\| : \mathcal{C}^n \to R$ is a norm if

1. $\|\vec{x}\| \geq 0$ with equality if and only if $\vec{x} = \vec{0}$
2. $\|\alpha\vec{x}\| = |\alpha| \ \|\vec{x}\|$
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

for all $\vec{x}, \vec{y} \in \mathcal{C}^n$ and all $\alpha \in \mathcal{C}$.

Let M_n denote the space of all n x n matrices over the complex field $\mathcal{C}$. A norm $\|\cdot\|$ on M_n is said to be a *matrix norm* if $\|\cdot\|$ is a norm and if, in addition, it satisfies

4. $\|AB\| \leq \|A\| \cdot \|B\|$

for all $A, B \in M_n$. We say a matrix norm $\|\cdot\|$ in the *operator norm* (induced by a vector norm $n(\cdot)$ on $\mathcal{C}^n$) if

$$\| A \| = \sup_{\vec{x} \in \mathcal{C}^n} \frac{n(A\vec{x})}{n(\vec{x})}$$

for all $A \in M_n$, and $\| \cdot \|$ is said to be a *weakly unitarily invariant norm* (w. u. i. n) if for any unitary matrix U,

$$\| U^* AU \| = \| A \|$$

for all $A \in M_n$, and a *unitarily invariant norm* (u. i. n) if for any unitary matrices U and V,

$$\| UAV \| = \| A \|$$

for all $A \in M_n$. We denote by $\| \cdot \|_2$ the operator norm induced by Euclidean norm on $\mathcal{C}^n$.

The reader who wishes for more information about w. u. i. n. and u. i. n. is recommended to read I. C. Gohberg, and M. G. Krein [4] and R. Schatten [7] .

Our single purpose in this paper is to give the condition that enables a w. u. i. n. to become $\| \cdot \|_2$.

For $\vec{x}, \vec{y} \in \mathcal{C}^n$, rank 1 matrix $\vec{x} \otimes \vec{y}$ is defined by

$$(\vec{x} \otimes \vec{y})\, \vec{z} := (\vec{z} \mid \vec{y})\, \vec{x}$$

for all $\vec{z} \in \mathcal{C}^n$. A norm $\| \cdot \|$ on M_n is called a *quasioperator norm* if there exists a vector norm $n(\cdot)$ on $\mathcal{C}^n$ such that

$$\| \vec{x} \otimes \vec{y} \| = n(\vec{x}) \cdot n^*(\vec{y}^*)$$

for all $\vec{x}, \vec{y} \in \mathcal{C}^n$, where $n^*(\cdot)$ is the dual norm of $n(\cdot)$; that is,

$$n^*(\vec{y}^*) = \sup_{\vec{x} \in \mathcal{C}^n} \frac{|(\vec{x} \mid \vec{y})|}{n(\vec{x})}$$

for all $\vec{y} \in \mathcal{C}^n$.

Our first result in as follows :

Theorem 1. Let $\| \cdot \|$ *be a quasioperator norm on* an M_n subordinate to the vector norm $n(\cdot)$. *If* $\| U^*(\vec{x} \otimes \vec{y}) U \| = \| \vec{x} \otimes \vec{y} \|$ *for all unitary matrices U and for all* $\vec{x}, \vec{y} \in \mathcal{C}^n$ *then* $n(\cdot)$ *is a scalar multiple of the Euclidean norm.*

Proof. For the unitary matrix U there exists a vector $\vec{y}$ such that

$$U\vec{y} = \xi \vec{y} \quad (|\xi| = 1)$$

Since $\| \cdot \|$ is a quasioperator norm and a weakly unitarily invariant norm for any vector $\vec{x}$, we have

$$\begin{aligned} n(\vec{x}) \cdot n^*(\vec{y}^*) &= \| \vec{x} \otimes \vec{y} \| \\ &= \| U^*(\vec{x} \otimes \vec{y}) U \| \\ &= \| U\vec{x} \otimes (U\vec{y})^* \| \\ &= n(U\vec{x}) \cdot n^*((U\vec{y})^*) \\ &= n(U\vec{x}) \cdot n^*((\xi\vec{y})^*) \\ &= n(U\vec{x}) \cdot n^*(\vec{y}^*), \end{aligned}$$

hence $n(\vec{x}) = n(U\vec{x})$ for any unitary matrix U and any vector $\vec{x}$.

Let $\vec{e} := {}^t(1, 0, \cdot, \cdot\cdot, 0)$. For vector $\vec{x}$, there exists a unitary matrix U such that

$$U\vec{x} = n_2(\vec{x})\vec{e}$$

where $n_2(\cdot)$ is the Euclidean norm on $\mathcal{C}^n$.

Therefore we have

$$n(\vec{x}) = n(U\vec{x}) = n(n_2(\vec{x})\vec{e}) = n(\vec{e})\cdot n_2(\vec{x})$$

and the proof of theorem is complete.

Theorem 2. Let $\|\cdot\|$ *be an operator norm on* M_n *. If* $\|\cdot\|$ *is a weakly unitarily invariant norm for all matrices of rank 1, then* $\|\cdot\|$ *is a Hilbert space operator norm.*

Proof. Since it is well known that an operator norm is a quasioperator norm (c. f. [2]), if $\|\cdot\|$ is subordinate to n (•), then, by Theorem 1, there is constant scalar $\alpha>0$ such that $n(\cdot) = \alpha\cdot n_2(\cdot)$.

Hence we have

$$\|A\| = \sup_{\vec{x}\neq 0}\frac{n(A\vec{x})}{n(\vec{x})} = \sup_{\vec{x}\neq 0}\frac{\alpha n_2(A\vec{x})}{\alpha n_2(\vec{x})} = \|A\|_2$$

for any $A \in M_n$

The next corollary easily follows.

Corollary 3. *Let* $\|\cdot\|$ *be an operator norm on* M_n *. Then* $\|\cdot\|$ *is unitarily invariant if and only if* $\|\cdot\|$ *is weakly unitarily invariant.*

If $\|\cdot\|$ is a matrix norm on M_n which is not an operator norm then corollary 3 may be false.

Example 4. Let $\|\cdot\|$ be a matrix norm which is not unitarily invariant. If we put $\|A\|' := \sup\{\|UAU^*\| \mid U ;\ \text{unitary matrix}\}$ then it is easy to see that $\|\cdot\|'$ is the matrix norm and that $\|\cdot\|'$ is weakly unitarily invariant. Hence we have $\|UA\|' = \|U^*(UA)U\| = \|AU\|'$ for all unitary U and for all $A \in M_n$. Now let $A = (a_{ij}) \in M_2$, and we define

$$\|A\| := \max(|a_{11}| + |a_{12}|,\ |a_{21}| + |a_{22}|)$$

If $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, then A is a unitary matrix, and

$$\|A\|' \geq \|A\| = \frac{2}{\sqrt{2}} > 1$$

On the other hand, if we put $U = A^*$, then $\|UA\|' = \|AU\|' = \|I\|' = 1$, and thus we obtain $\|A\|' > \|AU\|' = \|UA\|'$; hence, $\|\cdot\|'$ is not a unitarily invariant norm.

For vectors $\vec{x}$, $\vec{y}$, define the relation $\|$ by

$$\vec{x} \parallel \vec{y}^* :\Longleftrightarrow \|\vec{x}\|\cdot\|\vec{y}^*\|^* = (\vec{x} \mid \vec{y}) = 1,$$

which has been discussed by J. Stoer [9] and C. Zenger [10] .

J. Stoer showed that

$$\vec{x} \parallel \vec{y}^* \Rightarrow \| \vec{x} \otimes \vec{y} \|^* = 1$$

where $\| \cdot \|^*$ is the dual norm of $\| \cdot \|$, which is defined by

$$\| A \|^* := \sup_{\|B\|=1} | \operatorname{tr}(A \cdot B) | \quad \text{for all } A \in M_n.$$

C. Zenger remarked [10] that if $\vec{x} \parallel \vec{y}^*$ and $\vec{u} \parallel \vec{v}^*$ then

$$\| \vec{x} \otimes \vec{y} + \vec{u} \otimes \vec{v} \|_2^* = 2.$$

We will consider the inverse version in the terms

$$\left.\begin{array}{l} x \parallel y^*, \ \| \vec{u} \otimes \vec{v} \|_2 = 1 \geq | (\vec{u} \mid \vec{v}) | \\ (*) \cdots\cdots \| \vec{x} \otimes \vec{y} + \vec{u} \otimes \vec{v} \|_2^* = 2 \end{array}\right\} \underset{?}{\Rightarrow} \vec{u} \parallel \vec{v}^*.$$

The following gives a negative answer.

Example 5. By assumption,

$$2 = \| \vec{x} \otimes \vec{y} + \vec{u} \otimes \vec{v} \|_2^*$$

$$= \sup_{\|B\| \leq 1} | \operatorname{tr}(B(\vec{x} \otimes \vec{y} + \vec{u} \otimes \vec{v})) |$$

$$= \sup_{\|B\| \leq 1} | \operatorname{tr}(B\vec{x} \otimes \vec{y} + B\vec{u} \otimes \vec{v}) |.$$

Hence there exists matrix B_0 such that $\| B_0 \| \leq 1$ and

$$\begin{aligned} \| \vec{x} \otimes \vec{y} + \vec{u} \otimes \vec{v} \|_2^* &= \operatorname{tr}(B_0 \vec{x} \otimes \vec{y} + B_0 \vec{u} \otimes \vec{v}) \\ &= (B_o \vec{x} \mid \vec{y}) + (B_0 \vec{u} \mid \vec{v}) \\ &\leq | (B_0 \vec{x} \mid \vec{y}) | + | (B_0 \vec{u} \mid \vec{v}) | \\ &\leq \| \vec{x} \otimes \vec{y} \|_2^* + \| \vec{u} \otimes \vec{v} \|_2^* = 2. \end{aligned}$$

Since $| (B_0 \vec{x} \mid \vec{y}) | \leq \| \vec{x} \otimes \vec{y} \|_2^*$ and $| (B_0 \vec{u} \mid \vec{v}) | \leq \| \vec{u} \otimes \vec{v} \|_2^*$, we have

$$1 = | (B_0 \vec{x} \mid \vec{y}) | \leq \| B_0 \vec{x} \| \cdot \| \vec{y}^* \|^* \leq \| \vec{x} \| \cdot \| \vec{y}^* \|^* = 1,$$

therefore

$$| (B_0 \vec{x} \mid \vec{y}) | = \| B_0 \vec{x} \|_2 \cdot \| \vec{y}^* \|_2^* = 1.$$

Hence by the definition of relation $\parallel$ we obtain $B_0\vec{x} \parallel \vec{y}^*$.
As C. Zenger [10] has shown, the uniqueness of the relation $\vec{x} \parallel \vec{y}^*$ and $B_0\vec{x} \parallel \vec{y}^*$ imply $\vec{x} = B_0\vec{x}$.
Similarly we have $B_0\vec{u} \parallel \vec{v}$.
Therefore by the selection of B_0 and $\vec{u}$, $\vec{v}$ we have $| (\vec{u} \mid \vec{v}) | < 1 = \| \vec{u} \otimes \vec{v} \|_2^*$ but $(*)$ is satisfied.

2. Schur product

The Schur product of two nxn matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined by $A \circ B := (a_{ij} \cdot b_{ij})$. Recently the norm inequalities on the Schur product in the matrix space M_n have been studied (c. f. [1], [3], [5] and [6]).

For any $A \in M_n$, we denote by $c_1(A) \geq c_2(A) \geq \cdots\cdots \geq c_n(A) > 0$ the Euclidean lengths of the n columus of A, rearranged in decreasing order, by $r_1(A) \geq \cdots\cdots \geq r_n(A) > 0$ of the Euclidean lengths of n row of A , similarly ordered. The singular values of A which are listed in decreasing order are denoted by $s_1(A) \geq s_2(A) \cdots\cdots \geq s_n(A)$.

The near inequalities mentioned above are induced by the following result of T. Ando et. al. [1] .

Theorem 6. (T. Ando, R. A. Horn and C. R. Johnson [1].)
Let $A, B \in M_n$ *be given. Then*

$$\sum_{i=1}^{k} s_i(A \circ B) \leq \sum_{i=1}^{k} c_i(X) \cdot c_i(Y) \cdot s_i(B), \quad k = 1, \cdots\cdots, n$$

for any $X, Y \in M_n$ *such that* $A = X^*Y$.
For fixed $A \in M_n$, we define S_A by

$$S_A(B) := A \circ B \quad (B \in M_n).$$

It is easy to show that $S_A(\cdot)$ is a Linear operator on M_n , therefore we can consider the norm of S_A, that is,

$$\| S_A \| := \sup_{\|B\| \leq 1} \| S_A(B) \|_2 .$$

Since S. C. Ong [6] has proved that

$$\| S_A \| \leq \min \{c_1(A), r_1(A)\} ,$$

we shall now give another simple proof of this result produced in discussion with Nakamura.
I. Schur showed in [8] ,

$$A \geq O \text{ and } B \geq O \Rightarrow A \circ B \geq O$$

where $X \geq O$ means X is nonnegative definite.
Therefore for fixed $A \geq O$, S_A is a nonnegative linear operator on M_n.
Hence by using the well known Schur's result ([8]), we have,
$\| S_A \| = \| A \circ I \|_2 = \max(a_{11}, a_{22}, , \cdots\cdots , a_{nn})$
for $A = (a_{ij}) \geq O$.

For any $A \in M_n$, if we define

$$U := \begin{pmatrix} |A| & A^* \\ A & |A^*| \end{pmatrix}$$ where $|A| := (A^*A)^{1/2}$, then it is shown $U \geq O$

(c. f. [1] and [5]) hence we have

$$\| S_U \| = \| U \circ I \|_2 = \max_i (\max(p_{ii}, q_{ii}))$$

where $(p_{ij}) = |A|$ and $(q_{ij}) = |A^*|$.
On the other hand,

$$\| S_A \| = \sup_{\|B\| \leq 1} \| S_A(B) \|_2 = \sup_{\|B\| \leq 1} \| A \circ B \|_2 \leq \| S_U \|$$

is easily shown.
This concludes the proof of our claim.

Ong showed in [6] that if U is the unitary matrix then

$$\| S_U \| = 1 = \| S_{|U|} \| .$$

We will investigate the relation of $\| S_A \|$ and $\| S_{|A|} \|$ for some kinds of matrix.

Theorem 7. Let $A \in M_n$ *be normal. Then we have*

$$\| S_A \| \leq \| S_{|A|} \| .$$

Proof. If A is the normal matrix in M_n then by polar decomposion, there exist a unitary matrix U and a nonnegative matrix P such that $A = UP$ and $UP = PU$.
Since $UP = PU$, we have $A = P^{1/2} UP^{1/2}$.
Hence if we put $X^* = P^{1/2}$ and $Y = UP^{1/2}$ and then apply Theorem 5,

$$\| S_A \| \leq \sqrt{\max_1 (p_{11})} \cdot \sqrt{\max_1 (p_{11})} = \max_1 (p_{11}) = \| S_{|A|} \| .$$

The following example shows that equality can not put in a place of inequality in general even if A is a hermite matrix.

Example 8. Let A be a matrix $\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$ then by easy calculation, we have

$|A| = \frac{1}{\sqrt{17}} \begin{pmatrix} 10 & 6 \\ 6 & 7 \end{pmatrix}$. The preceding theorem shows $\| S_{|A|} \| = \frac{10}{\sqrt{17}} \fallingdotseq 2.427$.

Let us calculate $\| S_A \|$. Since the extreme points of the unit ball in M_n are the set of all unitary matrices, we have

$$\| S_A \| = \sup \{ \| A \circ U \|_2 \mid U : \text{unitary matrix} \} .$$

Therefor if $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ be a unitary matrix then we have

$A \circ U = \begin{pmatrix} 2u_{11} & 2u_{12} \\ 2u_{21} & u_{22} \end{pmatrix}$ and $|u_{i1}|^2 + |u_{i2}|^2 = 1 \ (i = 1, 2)$,

$u_{11}\bar{u}_{12} + u_{21}\bar{u}_{22} = 0$, $u_{11}\bar{u}_{21} + u_{12}\bar{u}_{22} = 0$ and
$|u_{1i}|^2 + |u_{2i}|^2 = 1 \ (i = 1, 2)$.
By using these results we can obtain $\| A \circ U \| \leq \sqrt{5} = 2.23\cdots\cdots$,
Consequently, since $\| S_A \| \leq 2.23\cdots\cdots$, we have $\| S_A \| < \| S_{|A|} \|$.

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