Discriminant Quadratic Forms and their Applications to the Classifications of Real K3 Surfaces

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ABSTRACT

In this note we introduce the notion of discriminant quadratic forms of lattices and apply it to the isometric classifications of integral involutions of the K3 lattice with some condition. Such an algebraic classification is the first step to the geometric investigations of real or complex K3 surfaces of some types. We give a detailed explanation of the enumeration of isometry classes of integral involutions of the K3 lattice with condition \((3, 1, 1), -\text{id}\). Each isometry class is given by a list of genus invariants, and some of these invariants are defined by discriminant quadratic forms of lattices.

1 Introduction

It is very important to investigate the intersection forms when we classify K3 surfaces not only topologically but also complex analytically. Intersection forms of compact complex surfaces are unimodular (see Definition 2.5) integral symmetric bilinear forms. Nikulin’s work [3] is a pioneer of studies of integral symmetric bilinear forms. We call nondegenerate (see Definition 2.1) integral symmetric bilinear forms lattices. In order to classify lattices or integral involutions of lattices, he introduced many genus (see Definition 2.7) invariants by using discriminant forms (see Definition 2.14) of lattices. His classification theories of lattices give strong tools for the classifications of K3 surfaces. For example, in [3], he completed the moduli classification of real polarized K3 surfaces. Moreover, Nikulin’s next paper [5] studied integral involutions of lattices with “conditions” (see Definition 3.3). His paper [5] also gave many applications to the studies of K3 surfaces. Especially, it was usefully applied to the classifications of 2-elementary K3 surfaces, Enriques surfaces, and real or complex lattice-polarized K3 surfaces (see [4], [1], [6] and many other articles).

This note starts from an easy introduction to discriminant forms of lattices. Finally we give a
detailed explanation of the enumeration of all the isometry classes of integral involutions of the K3 lattice with condition \((3, 1, 1), -\text{id}\).

## 2 Lattices and their discriminant forms

### 2.1 Integral symmetric bilinear forms and lattices

Let \( S \) be a free \( \mathbb{Z} \)-module of finite rank \( s \) \((\geq 1)\), and \( b : S \times S \to \mathbb{Z} \) a symmetric bilinear form over \( \mathbb{Z} \), where \( \mathbb{Z} \) is the ring of rational integers.

**Definition 2.1 (Nondegenerate forms)** We say a symmetric bilinear form \( b \) (or \( S \)) is nondegenerate if the natural map \( i : S \to S^* := \text{Hom}(S, \mathbb{Z}) \), where \( i(x)(y) = x \cdot y \) for \( x, y \in S \), is injective.

**Remark 2.2** \( S \) is nondegenerate if and only if \( i(S) \) is also a (free) \( \mathbb{Z} \)-module of the same rank \( s \). We always have \( S^* := \text{Hom}(S, \mathbb{Z}) \cong \mathbb{Z}^s \). If \( S \) is nondegenerate, then we can choose a basis \( \{ v_i \mid i = 1, \ldots, s \} \) of \( S^* \) such that \( \theta_1 v_1, \ldots, \theta_s v_s \) is a basis of \( i(S) \) for some positive integers \( \theta_1, \ldots, \theta_s \). Then the quotient group \( S^*/i(S) \) is isomorphic to the finite abelian group \((\mathbb{Z}/\theta_1 \mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/\theta_s \mathbb{Z}) \). Conversely, if \( S^*/i(S) \) is a finite abelian group, then \( i(S) \) is a (free) \( \mathbb{Z} \)-module of rank \( s \).

**Definition 2.3 (Lattices)** By a lattice we mean a free \( \mathbb{Z} \)-module of finite rank with a nondegenerate symmetric bilinear form with values in \( \mathbb{Z} \).

**Definition 2.4 (Parity)** A lattice \( S \) is called even if \( x^2 = x \cdot x \) \((\in \mathbb{Z})\) is even for all \( x \in S \), and odd otherwise.

**Definition 2.5 (Unimodular lattices)** The discriminant of a lattice \( S \) is defined to be \( \text{discr} S := \det(e_i : e_j) \), where \( \{ e_i \} \) is some basis of \( S \). The discriminant of every lattice is a nonzero integer.

We say a lattice \( S \) is unimodular if \( \text{discr} S = \pm 1 \), namely, the matrix \( \{ e_i : e_j \} \) is invertible.

By \( S_1 \oplus S_2 \) we denote the orthogonal direct sum of lattices \( S_1 \) and \( S_2 \).

For a lattice \( S \), we denote by \( S(m) \) the lattice obtained from \( S \) by multiplying the form of \( S \) by \( m \in \mathbb{Q} \) assuming that \( S(m) \) is also an integral lattice. Here \( \mathbb{Q} \) is the field of rational numbers.

We denote by \( (m) \) the lattice of rank 1 whose generator \( x \) satisfies \( x^2 = m(\neq 0) \in \mathbb{Z} \). For example, we use the lattice \((-2) \) later.

**Definition 2.6 (Isometry classes over \( \mathbb{Z} \) and \( \mathbb{Z}_p \))** Let \( S, T \) be free \( \mathbb{Z} \)-modules of finite rank, and \( b_S : S \times S \to \mathbb{Z} \), \( b_T : T \times T \to \mathbb{Z} \) be symmetric bilinear forms. We say \( b_S \) and \( b_T \) are isometric if there exists an isomorphism \( f : S \to T \) such that \( b_S(x, y) = b_T(f(x), f(y)) \) for all \( x, y \in S \).

Let \( \mathbb{Z}_p \) be the ring of \( p \)-adic integers. If \( S, T \) are free \( \mathbb{Z}_p \)-modules of finite rank and \( b_S : S \times S \to \mathbb{Z}_p \), \( b_T : T \times T \to \mathbb{Z}_p \) are symmetric bilinear forms over \( \mathbb{Z}_p \), then we say \( b_S \) and \( b_T \) are isometric if there exists a \( \mathbb{Z}_p \)-isomorphism \( f : S \to T \) such that \( b_S(x, y) = b_T(f(x), f(y)) \) for all \( x, y \in S \).

**Definition 2.7 (Genera, cf. [2])** We say symmetric bilinear form \( b_S : S \times S \to \mathbb{Z} \) and \( b_T : T \times T \to \mathbb{Z} \) belong to the same genus if induced bilinear forms \( b_S : \mathbb{Z}_p \otimes \mathbb{Z}_p \times S \otimes \mathbb{Z}_p \to \mathbb{Z}_p \) and \( b_T : T \otimes \mathbb{Z}_p \times T \otimes \mathbb{Z}_p \to \mathbb{Z}_p \) over \( \mathbb{Z}_p \) are isometric for every prime \( p = 2, 3, 5, 7, \ldots \) and \( p = \infty \). Here \( \mathbb{Z}_p \) stands for the ring of \( p \)-adic integers, and \( \mathbb{Z}_\infty := \mathbb{R} \).

There is a famous result for unimodular lattices:

**Theorem 2.8 (J.Milnor, cf. [10])**

(a) The parity (even or odd) and signature \( (l_+, l_-) \) uniquely determine the genus (see Definition 2.7 above), and if \( l_+ > 0 \) and \( l_- > 0 \) \((i.e., \text{indefinite case})\), then they also determine the isometry class of a given unimodular lattice.

(b) An even unimodular lattice of signature \( (l_+, l_-) \) exists if and only if

\[ l_+ - l_- \equiv 0 \pmod{8}, \]

where \( l_+ \) and \( l_- \) are nonnegative integers.

We use the symbols \( \U, \E_8 \), and \( L_{K3} \).
respectively, to denote the even unimodular lattices of signature \((1, 1), (0, 8), (3, 19)\). By Theorem 2.8, \(U, L_{K3}\), respectively, are uniquely determined up to isometry. It is known that \(E_8\) is also unique up to isometry. Especially, we have \(L_{K3} \cong U \oplus U \oplus E_8 \oplus E_8\). Sometimes we call \(L_{K3}\) the \(K3\) lattice.

**Definition 2.9 (Primitive embeddings)** An embedding of lattices (i.e., injective homomorphism preserving the forms) \(p : S \rightarrow M\) is called primitive if \(M/p(S)\) is a free \(\mathbb{Z}\)-module.

Two primitive embeddings \(p : S \rightarrow M\) and \(p' : S \rightarrow M'\) is called isometric if there exists an isometry \(\varphi : M \rightarrow M'\) reducing to the identity map on \(S\), that is, \(\varphi \circ p = p'\).

(If there exists an isometry \(\varphi : M \rightarrow M'\) taking \(p(S)\) to \(p'(S)\), then \(p\) and \(p' \circ \varphi\) are isometric, where we set \(\psi := (p')^{-1} \circ \varphi \circ p : S \rightarrow S\).)

### 2.2 Finite symmetric bilinear forms and finite quadratic forms.

**Definition 2.10 (Finite forms)** By a finite symmetric bilinear form we mean a symmetric bilinear form

\[b : A \times A \rightarrow \mathbb{Q}/\mathbb{Z}\]

defined on a finite abelian group \(A\).

By a finite quadratic form we mean a map

\[q : A \rightarrow \mathbb{Q}/2\mathbb{Z}\]

satisfying the following conditions:

1) \(q(na) = n^2 q(a)\) for all \(n \in \mathbb{Z}\) and \(a \in A\).

2) \(q(a + a') - q(a) - q(a') \equiv 2b(a, a') \pmod{2\mathbb{Z}}\), where \(b : A \times A \rightarrow \mathbb{Q}/\mathbb{Z}\) is an indefinite symmetric bilinear form, which we call the associated bilinear form of \(q\).

**Definition 2.11** We say a finite quadratic form \(q\) is nondegenerate if the natural map

\[i_A : A \rightarrow \text{Hom}(A, \mathbb{Q}/\mathbb{Z})\]

is injective, where \(i_A(x)(y) := b(x, y)\) for \(x, y \in A\) and \(b\) is the associated bilinear form of \(q\).

We introduce the notion of orthogonality (\(\perp\)) on subgroups of a finite abelian group \(A\) by means of the bilinear form \(b\) on \(A\), and also the notion of orthogonal sum (\(\oplus\)) of finite symmetric bilinear forms and quadratic forms.

### 2.3 Discriminant bilinear forms and discriminant quadratic forms of lattices

Let \(S\) be a lattice. The following lemma is well-known:

**Lemma 2.12** We define a map \(j : S^* := \text{Hom}(S, \mathbb{Z}) \rightarrow S \otimes \mathbb{Q}\) as follows. For \(x \in S^*\), there exists an integer \(p\) such that \(px \in i(S)\). Then, there exists \(\varepsilon \in S\) such that \((px)(u) = \varepsilon \cdot u\) for all \(u \in S\). We set

\[j(x) := \varepsilon \otimes \frac{1}{p}\]

Then \(j\) is well-defined and injective.

**Proof.** (1) Suppose that there exists another integer \(p'\) such that \(p'x \in i(S)\) and there exists \(\varepsilon' \in S\) such that \((p'x)(u) = \varepsilon' \cdot u\) for all \(u \in S\). Then \((p' \varepsilon) \cdot u = p' \varepsilon' \cdot u\) for all \(u \in S\). Since \(S\) is nondegenerate, we have \(p' \varepsilon = p' \varepsilon'\). Thus, \(\varepsilon \otimes \frac{1}{p} = (p' \varepsilon) \otimes \frac{1}{p'p} = (p' \varepsilon') \otimes \frac{1}{p'p} = \varepsilon' \otimes \frac{1}{p}\). Thus, \(j\) is well-defined. (2) For \(y \in S^*\), suppose that there exists an integers \(q\) such that \(qy \in i(S)\) and there exists \(\delta \in S\) such that \((qy)(u) = \delta \cdot u\) for all \(u \in S\). If \(j(x) = j(y)\), namely, \(\varepsilon \otimes \frac{1}{p} = \delta \otimes \frac{1}{q}\), then we have \(q \varepsilon p = \delta q\). Hence, \(p q y)(u) = p \delta \cdot u = q \varepsilon \cdot u = (q p x)(u)\) for all \(u \in S\). Hence, \(x = y\). This means that \(j\) is injective. It is easy to prove \(j\) is a \(\mathbb{Z}\)-module homomorphism. Thus, \(j : S^* \rightarrow S \otimes \mathbb{Q}\) is injective.

Then we can extend the bilinear form \(b_S\) on \(S\) to one on \(S^* \otimes S \otimes \mathbb{Q}\) taking values in \(\mathbb{Q}\) in the natural way. For \(x, y, S \in S^*\), there exist integers \(p, q, r\) such that \(p x, q y \in i(S)\). Then, there exist \(\varepsilon\) and \(\delta\) in \(S\) such that \((p x)(u) = \varepsilon \cdot u\) and \((q y)(u) = \delta \cdot u\) for all \(u \in S\). Then we set

\[b_S(x, y) := j(x) \cdot j(y) = (\varepsilon \otimes \frac{1}{p}) \cdot (\delta \otimes \frac{1}{q}) = \frac{1}{pq} (\varepsilon \cdot \delta) = \frac{1}{q} x(\delta) = \frac{1}{p} y(\varepsilon) \quad (\in \mathbb{Q}).\]
Definition 2.13 (Discriminant groups) For a lattice $S$, we call the quotient group $A_S := S^*/i(S)$ the discriminant group of $S$.

Definition 2.14 (Discriminant bilinear forms of lattices) We define a finite symmetric bilinear form $b_S : A_S \times A_S \to \mathbb{Q}/\mathbb{Z}$ by

$$b_S([x], [y]) := [b_S(x, y)] \in \mathbb{Q}/\mathbb{Z}.$$ 

We call $b_S$ the discriminant bilinear form of a lattice $S$.

The above definition is well-defined. Indeed, if $[x] = [x']$ and $[y] = [y']$, then there exist $\alpha$ and $\beta$ in $S$ such that $(x - x')(u) = \alpha \cdot u$ and $(y - y')(u) = \beta \cdot u$ for all $u \in S$. Then, by the above definition, we have $b_S(x, y) = \frac{1}{pq} \epsilon \cdot \delta = \frac{1}{q} \epsilon(\delta) = \frac{1}{q} \frac{1}{p} \epsilon' \cdot \delta + \alpha \cdot \delta = \frac{1}{q} \frac{1}{p} \epsilon' \cdot \delta + \frac{1}{q} \beta \cdot \epsilon + y(\alpha) = \frac{1}{q} \frac{1}{p} (y(\epsilon') \cdot \beta + \epsilon' \cdot \alpha + y(\alpha)) = \frac{1}{q} \frac{1}{p} \epsilon' \cdot \beta + x'(\beta) + y(\alpha) = b_S(x', y') \in \mathbb{Q}/\mathbb{Z}$. And hence $[b_S(x, y)] = [b_S(x', y')]$ in $\mathbb{Q}/\mathbb{Z}$. □

Definition 2.15 (Discriminant quadratic forms of lattices) If $S$ is even, then we define $q_S : A_S \to \mathbb{Q}/2\mathbb{Z}$ by

$$q_S([x]) := [b_S(x, x)] \in \mathbb{Q}/2\mathbb{Z}.$$ 

We call $q_S$ the discriminant quadratic form of a lattice $S$.

The above definition is well-defined. Indeed, if $[x] = [x']$, then there exist $\alpha$ in $S$ such that $(x - x')(u) = \alpha \cdot u$. Hence, $b_S(x, x) = b_S(x', x') + x'(\alpha) + x(\alpha) = b_S(x', x') + x'(\alpha) + x'(\alpha) + (x'(\alpha) + x(\alpha)) = b_S(x', x') + 2x'(\alpha) + \alpha \cdot \alpha$. Thus, $b_S(x, x) = b_S(x', x')$ in $\mathbb{Q}/2\mathbb{Z}$. □

It is easily seen that $q_S$ is a finite quadratic form and the associated bilinear form of $q_S$ is $b_S$.

Remark 2.16 The discriminant quadratic form $q_S$ of a lattice $S$ is nondegenerate. Actually, let us consider $i_{A_S} : A_S \to \text{Hom}(A_S, \mathbb{Q}/\mathbb{Z})$, where $i_{A_S}([x])([y]) = b_S([x], [y])$ for $[x], [y] \in A_S$. Suppose that $i_{A_S}([x]) = 0$, namely, $i_{A_S}([x])([y]) = b_S([x], [y]) = [0]$ for all $[y] \in A_S$. Then $b_S(x, y) = j(x) \cdot j(y) \in \mathbb{Z}$ for all $y \in S^*$. Recall that $j(x) = \epsilon \otimes \frac{1}{p}$ and $j(y) = \delta \otimes \frac{1}{q}$. Assume that $\epsilon = \epsilon'p'$ and $\epsilon'$ is a primitive element in $S$. There exists an element $z \in S^*$ such that $z(\epsilon') = 1$. Then $z(\epsilon) = p'$. We set $y := z$. Then $q_S(\epsilon) = \delta \cdot \epsilon$, and we have $q_S(\epsilon) = \delta \otimes \frac{1}{p}$. Then $j(x) \cdot j(y) = \epsilon \otimes \frac{1}{p} \delta \otimes \frac{1}{q} = \frac{p}{q} \in \mathbb{Z}$, and hence, $p$ divides $p'$. Hence, for all $u \in S$, $p(\epsilon) = \epsilon \cdot u = \epsilon \cdot u \cdot p' \epsilon'$, namely, $x(\epsilon) = \left(\frac{\epsilon}{p'} \epsilon'ight) \cdot u$. Here $\epsilon \epsilon' \in S$. Hence, $x \in i(S)$ and $[x] = 0$. Thus $i_{A_S}$ is injective. □

3 Integral involutions of lattices

3.1 Integral involutions of the lattice $\mathbb{L}_{K_3}$ with condition $(S, 0)$

For a lattice $S$, we set $r(S) := \text{rank} S$.

Definition 3.1 We say a lattice $S$ is $2$-elementary if its discriminant group $A_S := S^*/i(S)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\alpha(S)}$ for some non-negative integer $\alpha(S)$. Then the non-negative integer $\alpha(S)$ is defined.

We say a lattice $S$ is hyperbolic (or Lorentzian) if the signature of $S$ is $(1, r(S) - 1)$.

In the sequel we write $S'/S$ instead of $S^*/i(S)$ omitting $i$.

Recall that $\mathbb{L}_{K_3}$ stands for the even unimodular lattice of signature $(3, 19)$ (the K3 lattice).

Let $S$ be a hyperbolic $2$-elementary primitive sublattice of $\mathbb{L}_{K_3}$. We define the "parity" $\delta(S)$ of $S$ by

$$\delta(S) := \begin{cases} 0 & \text{if } x^2 \in \mathbb{Z} \text{ for all } x \in S^*, \\ 1 & \text{otherwise}. \end{cases}$$

Equivalently, we may also define $\delta(S)$ by

$$\delta(S) := \begin{cases} 0 & \text{if } z \cdot \tau(z) \equiv 0 \pmod{2} \text{ for all } z \in \mathbb{L}_{K_3}, \\ 1 & \text{otherwise}, \end{cases}$$

where $\tau : \mathbb{L}_{K_3} \to \mathbb{L}_{K_3}$ is the unique integral involution which satisfies $S = \mathbb{L}_{K_3}^\tau$ (the fixed part of $\tau$).
Here, by an integral involution $\tau$ of a lattice $L$ we mean an automorphism with $\tau \circ \tau = \text{id}_L$.

We can say that $\delta(S)$ is the invariant of the integral involution $\tau$ of $\mathbb{L}_{K_3}$.

**Remark 3.2** ([4], [1]) Let $S$ be a primitive hyperbolic 2-elementary sublattice of $\mathbb{L}_{K_3}$. Then the triplet $(\tau(S), a(S), \delta(S))$ determines the isometry class of $S$. Moreover, if $S, S'$ are two primitive hyperbolic 2-elementary sublattices of the $K_3$ lattice $\mathbb{L}_{K_3}$ and these lattices are isometric, then there exists an automorphism $f$ of $\mathbb{L}_{K_3}$ such that $f(S') = S$ ([4], [1], p.124).

Let $S$ be a primitive hyperbolic 2-elementary sublattice of $\mathbb{L}_{K_3}$, and

$$\theta : S \rightarrow S$$

be an integral involution, i.e., an automorphism with $\theta \circ \theta = \text{id}_S$.

**Definition 3.3** (Conditions, [5]) By a condition on an integral involution of the lattice $\mathbb{L}_{K_3}$ we mean a triple

$$S := (S, \theta).$$

We denote $S_+ := S^0$ (the fixed part of $\theta$) and $S_- := S_0$ (the anti-fixed part of $\theta$), and set $p := \text{rank } S_+.$

We assume $S_+$ has the signature $(0, p)$ (negative definite).

All the genus invariants for a condition $(S, \theta)$ were found in [5] together with necessary and sufficient conditions of the existence.

**Definition 3.4** ([5], [8]) By an integral involution of the lattice $\mathbb{L}_{K_3}$ with condition $S = (S, \theta)$ we mean an integral involution $\psi$ of $\mathbb{L}_{K_3}$ compatible with the action of $\theta$ on $S$ and of $\psi$ on $\mathbb{L}_{K_3}$, that is, $\psi \circ i = i \circ \theta$, namely, the following diagram commutes:

$$
\begin{array}{ccc}
S & \rightarrow & \mathbb{L}_{K_3} \\
\theta \downarrow & & \downarrow \psi \\
S & \rightarrow & \mathbb{L}_{K_3},
\end{array}
$$

where $i : S \rightarrow \mathbb{L}_{K_3}$ is the inclusion map. It means that $\psi(x) = \theta(x)$ for every $x \in S$. (Then we have $\psi(S) = S$. )

**Remark 3.5** In [6], [7], [8] and [9], an integral involution of the lattice $\mathbb{L}_{K_3}$ with condition $S = (S, \theta)$ is called an integral involution of $\mathbb{L}_{K_3}$ of "type" $(S, \theta)$.

To define isometries between two integral involutions of the lattice $\mathbb{L}_{K_3}$ with condition $(S, \theta)$, we introduce a subgroup $G$ of $O(S, \theta) := \{f \in O(S) \mid f \circ \theta = \theta \circ f\}$. However, following [6], we have the following lemma.

**Lemma 3.6** ([6]) If $r(S) + a(S) < 20$ (in particular, if $r(S) < 10$), then the group $G = W(-4)(S_-, S) \times W(-4)(S_+, S)$.

Here we define $W(-4)(S_\pm, S)$ as follows. We set $\Delta(S_\pm, S)^{(-4)} = \{\alpha \in S_\pm \mid \alpha^2 = -4, \alpha \cdot S \equiv 0 \text{ mod } 2\}$. Let

$$W(-4)(S_\pm, S)$$

be the groups generated by reflections in all elements of $\Delta(S_\pm, S)^{(-4)}$.

**Definition 3.7** (Isometries with respect to $G$, [5], [6]) Two integral involutions $\psi$ and $\psi'$ of the lattice $\mathbb{L}_{K_3}$ with condition $S = (S, \theta)$ are called isometric with respect to $G$ if there is an isometry $f : \mathbb{L}_{K_3} \rightarrow \mathbb{L}_{K_3}$ between lattices with integral involutions, that is, $\psi' \circ f = f \circ \psi$ such that $f$ preserves the condition $S$, that is, $f \circ i = i \circ g$ for some $g \in G$. This means that $f(S) = S$ and $f|_{S} \in G$.

**Definition 3.8** (Genera, [5], [6]) We say that two integral involutions $\psi$ and $\psi'$ of type $(S, \theta)$ have the same genus with respect to $G$ if there exists an automorphism $\xi : S \rightarrow S$ from $G$ which can be continued to an isomorphism $(\mathbb{L}_{K_3}, \psi) \otimes \mathbb{R} \rightarrow (\mathbb{L}_{K_3}, \psi') \otimes \mathbb{R}$ over $\mathbb{R}$, and an isomorphism $(\mathbb{L}_{K_3}, \psi') \otimes \mathbb{Z}_p \rightarrow (\mathbb{L}_{K_3}, \psi') \otimes \mathbb{Z}_p$ over the ring $\mathbb{Z}_p$ of $p$-adic integers for any prime $p$. 

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3.2 Genus invariants of integral involutions of the lattice $L_{K3}$ with condition $(S, \theta)$

We describe the genus invariants of an integral involution $\psi$ of type $(S, \theta)$. To simplify notations, we denote

$$L_{K3}^+ := L_{K3}^\psi := \{ x \in L_{K3} \mid \psi(x) = x \}$$

and

$$L_{K3}^- := L_{K3,\psi} := \{ x \in L_{K3} \mid \psi(x) = -x \}.$$

We assume that the fixed part $L_{K3,\psi}$ of $\psi$ is of signature $(1, t_{(-)})$ (hyperbolic). We set

$$r := \text{rank} L_{K3}^\psi(= 1 + t_{(-)}).$$

Since $L_{K3}$ is unimodular, we have

$$A_{L_{K3}} = L_{K3}^+/L_{K3} \cong L_{K3}/(L_{K3}^+ \oplus L_{K3}^-) \cong (\mathbb{Z}/2\mathbb{Z})^a,$$

for some nonnegative integer $a$. We define

$$\delta_{\psi} := \begin{cases} 0 & \text{if } x \cdot \psi(x) \equiv 0 \mod 2 \text{ for all } x \in L_{K3} \\
1 & \text{otherwise.} \end{cases}$$

Note that we always have some element $s_{\psi} \in L_{K3}$ which satisfies

$$x \cdot \psi(x) \equiv x \cdot s_{\psi} \mod 2 \text{ for all } x \in L_{K3}.$$

These elements $s_{\psi}$ are unique modulo $2L_{K3}$ and called the characteristic elements of an integral involution $\psi$ of $L_{K3}$. Note that $\delta_{\psi} = 0$ means that we can take $0$ as the characteristic element $s_{\psi}$. We define

$$\delta_{\psi,S} := \begin{cases} 0 & \text{if } x \cdot \psi(x) \equiv x \cdot s_{\psi} \mod 2 \text{ for all } x \in L_{K3} \\
& \text{for some element } s_{\psi} \text{ in } S, \\
1 & \text{otherwise.} \end{cases}$$

Then $\delta_{\psi,S} = 0$ means that we can find $s_{\psi}$ exactly in $S$.

Depending on these invariants, we divide integral involutions $\psi$ into the following 3 types:

**Type 0:** $\delta_{\psi,S} = 0$ and $\delta_{\psi} = 0$,

**Type Ia:** $\delta_{\psi,S} = 0$ and $\delta_{\psi} = 1$, and

**Type Ib:** $\delta_{\psi,S} = 1$.

We now quote the formulations and the definitions from [5] and [8].

For each $x_{\pm} \in S_{\pm}$ we define

$$\delta_{x_{\pm}} := \begin{cases} 0 & \text{if } x_{\pm} \cdot L_{K3} \equiv 0 \mod 2, \\
1 & \text{otherwise.} \end{cases}$$

Since $L_{K3}$ is unimodular, $\delta_{x_{\pm}} = 0$ if and only if there exists $x_{\pm}' \in L_{K3}^{\pm}$ such that $\frac{1}{2}(x_{\pm} + x_{\pm}') \in L_{K3}$. The elements $x_{\pm}'$ are defined by the elements $x_{\pm}$ uniquely modulo $2(L_{K3}^{\mp})$. Thus, for elements $x_{+} \in S_{+}$ and $x_{-} \in S_{-}$ satisfying $\delta_{x_{+}} = \delta_{x_{-}} = 0$, we define the invariant

$$\rho_{x_{+},x_{-}} := \frac{1}{2} x_{+} \cdot x_{-}' \mod 2 = -\frac{1}{2} x_{+}' \cdot x_{-} \mod 2 \in (\mathbb{Z}/2\mathbb{Z}).$$

Thus, we get a list of genus invariants

$$(r, a; \delta_{x_{+}}, \delta_{x_{-}}, \rho_{x_{+},x_{-}}, \delta_{\psi}, \delta_{\psi,S}, s_{\psi}),$$

where the characteristic element $s_{\psi}$ in $S$ mod $2S$ is defined if $\delta_{\psi,S} = 0$; and the invariant $\rho_{x_{+},x_{-}}$ is defined only if $\delta_{x_{+}} = \delta_{x_{-}} = 0$. 


Proposition 3.9 ([5]) Two integral involutions with condition \((S, 0)\) have the same genus with respect to \(G\) if and only if the invariants \((3.1)\) are conjugate by the group \(G\).

To formulate (see Theorem 3.11) the conditions of existence of an integral involution with the genus invariants \((3.1)\), we need to reformulate the invariants \((3.1)\) more. We have the function \(\delta_\pm : S_\pm \to \mathbb{Z}/2\mathbb{Z}\) where \(x_\pm \to \delta_\pm\), and we set

\[ H_\pm := \delta_\pm^{-1}(0)/2S_\pm \subset S_\pm/2S_\pm. \]

These subgroups \(H_\pm\) are equivalent to the invariants \(\delta_\pm\). For \(H_\pm\), we have

\[ \Gamma_\pm := (2S)_\pm/2S_\pm \subset H_\pm \subset 2(S_\pm \cap (\frac{1}{2}S_\pm))/2S_\pm \cong (S_\pm \cap (\frac{1}{2}S_\pm))/S_\pm = A_{S_\pm}^{(2)} \subset A_{S_\pm}, \]

where \((2S)_\pm\) denote the orthogonal projections of \(2S \subset S_+ \oplus S_-\) to \(S_\pm\) respectively. This projections also give the graph \(\Gamma\) of the isomorphism \(\gamma\) of the groups \(\Gamma_+\) and \(\Gamma_-\). \(A_{S_\pm}^{(2)}\) denotes the subgroup of \(A_{S_\pm}\) generated by all elements of order 2 respectively. Let

\[ H := H_+ \oplus \gamma H_- := (H_+ \oplus H_-)/\Gamma. \]

For simplicity we identify \(H_\pm\) and \(H_\pm\ mod \ \Gamma \subset H\) respectively.

We can define a finite quadratic form

\[ q_\rho : H \to \frac{1}{2}\mathbb{Z}/2\mathbb{Z} \]

by \(q_\rho|H_+ = q_{S_+}|H_+; \quad q_\rho|H_- = -q_{S_-}|H_-; \quad q_\rho(x_+, x_-) = \rho_{x_+, x_-} \mod \mathbb{Z}\) for \(x_\pm \in H_\pm\). The finite quadratic form \(q_\rho\) is equivalent to the invariants \(\rho_{x_+, x_-}\). In \(H_+ \times \Gamma_+\) and \(\Gamma_+ \times H_-\) the invariant \(\rho_{x_+, x_-}\) is defined by the discriminant quadratic forms \(q_{S_+}\) and \(q_{S_-}\) respectively. We define

\[ \rho : H_+ \times H_- \to \mathbb{Z}/2\mathbb{Z} \]

by \(\rho(x_+, x_-) := \rho_{x_+, x_-}\). Then note that \(q_\rho(x_+, x_-) = \rho(x_+, x_-)/2 \mod 1\).

If \(\delta_\psi S = 0\), we define

\[ v := s_\psi \in H = H_+ \oplus \gamma H_- \subset (S_+ \oplus S_-)/2S. \]

Then the element \(v\) is zero if \(\delta_\psi = 0\), and \(v\) is not zero if \(\delta_\psi = 1\). Note that \(v\) is a characteristic for the finite quadratic form \(q_\rho\) on \(H\), that is, \(q_\rho(x, v) \equiv q_\rho(x, x)/2 \mod 1\) for all \(x \in H\).

Then it is concluded that the genus invariants \((3.1)\) are equivalent to the genus invariants

\[ (r, \alpha; H_+, H_-, q_\rho, \delta_\psi, \delta_\psi S, v). \quad (3.2) \]

Finally we introduce some numerical invariants for the data \((3.2)\) which are important for their existence.

Lemma 3.10 ([5], [8]) Any finite 2-elementary quadratic form is an orthogonal sum \(\oplus\) of some elementary forms, where all the elementary forms are:

- \(z\) on \(\mathbb{Z}/2\mathbb{Z}\) with \(z(\xi) = 0 \mod 2\);
- \(w\) on \(\mathbb{Z}/2\mathbb{Z}\) with \(w(\xi) = 1 \mod 2\);
- \(u_+(2)\) on \(\mathbb{Z}/2\mathbb{Z}\) with \(u_+(\xi_1) = u_+(\xi_2) = 0 \mod 2, \quad u_+(\xi_1, \xi_2) = \frac{1}{2} \mod 1, \quad \sigma(u_+(2)) \equiv 0 \mod 8;\)
- \(v_+(2)\) on \(\mathbb{Z}/2\mathbb{Z}\) with \(v_+(\xi_1) = v_+(\xi_2) = 1 \mod 2, \quad v_+(\xi_1, \xi_2) = \frac{1}{2} \mod 1, \quad \sigma(v_+(2)) \equiv 4 \mod 8;\)
- \(q_\alpha(2)\) on \(\mathbb{Z}/2\mathbb{Z}\) with \(q_\alpha(2)(\xi) = \alpha/2 \mod 2, \quad \text{where} \ \alpha = \pm 1 \mod 4, \quad \sigma(q_\alpha(2)) \equiv 1 \mod 8 \text{ by the definition}. \)

Then any finite 2-elementary quadratic form \(f\) with a non-degenerate bilinear form is an orthogonal sum of \(u_+(2), v_+(2)\) and \(q_\alpha(2)\), and it has the invariant \(\sigma\) mod 8 which is equal to the sum of \(\sigma\) for all its summands. Moreover, it has an additional invariant \(\delta\). We define \(\delta = 0\) if \(f\) is the orthogonal sum of only \(u_+(2)\) and \(v_+(2)\), equivalently, \(f\) takes values in \(\mathbb{Z}\) mod 2. Otherwise, we define \(\delta = 1\).

Up to isomorphisms, \(f\) is defined by its rank \(k, \sigma\) mod 8 and \(\delta\) (\(= 0\) or 1). Then we write \(f = q(k, \delta, \sigma)\).

Let \(a_M\) be the rank over \(\mathbb{Z}/2\mathbb{Z}\) of a 2-elementary group \(M\). We have the numerical invariants \(a_{H_\pm}, a_{\Gamma_\pm} = a_{H_+} = a_{H_-} = a_{\Gamma_\pm}\).
The quadratic form \( q_\rho \) on \( H \) is one of forms of rank \( a_H = a_{H_+} + a_{H_-} - a_{\Gamma_+} \) with the invariants 
\[ \delta_H = (0 \text{ or } 1), \mu_\rho = (0 \text{ or } 1): \]
\[ \delta_H = 0: q_\rho = z^{k_\rho} u^{\mu_{\rho}} w^{\nu_{\rho}} u_{\pm} (2)^{\sigma_{\rho}/4} \otimes u_{\pm} (2)^{(a_H - k_\rho)/2 - \sigma_{\rho}/4}, \]
where \( \mu_\rho = 0 \text{ or } 1, \sigma_{\rho} \equiv 0 \text{ or } 4 \mod 8, \text{ and } \sigma_{\rho} \geq 0, \text{ and } \mu_\rho + \sigma_{\rho}/4 \leq 1; \]
\[ \delta_H = 1, \mu_\rho = 0: q_\rho = z^{k_\rho} u^{a_H - k_\rho}, \text{ and } \sigma_{\rho} \geq 0, \text{ and } \mu_\rho + \sigma_{\rho}/4 \leq 1; \]
\[ \delta_H = 1, \mu_\rho = 1: q_\rho = z^{k_\rho - 1} u^{a_H - k_\rho}, \text{ where } a_H > k_\rho. \]

The invariant \( \delta_H = 0 \) if and only if \( q_\rho(x) \equiv 2 \mod 2 \) for any \( x \in H \). We similarly introduce the invariants \( \delta_{H_\pm} \) for the form \( q_\rho|_{H_\pm} \). Then we have \( \delta_H = \max\{\delta_{H_+}, \delta_{H_-}\} \).

If \( \delta_\psi = 0 \) and \( v := s_\psi \in H \), then we get the invariant 
\[ c_v \pmod 4, \text{ where } c_v/2 = q_\rho(v) \pmod 2. \]

Thus we get the numerical invariants
\[ (a_{H_\pm}, a_{\Gamma_+}, \delta_{H_\pm}, k_\rho, \mu_\rho, \sigma_\rho, c_v) \quad (3.3) \]
for the data (3.2).

We need some more invariants for the data (3.2). An element \( \bar{v} \in A_{S_\pm}^{(2)} \) is called characteristic if 
\[ q_{S_\pm}(\bar{v}, x) \equiv 0 \mod 1 \text{ for any } x \in A_{S_\pm}^{(2)}. \]

We define \( \delta_\psi S_\pm = 0 \) if \( \delta_\psi = 0 \), the element \( v := s_\psi \in H_\pm \), and \( v \) is equal to the characteristic element of \( A_{S_\pm}^{(2)} \). Otherwise, we define \( \delta_\psi S_\pm = 1 \).

Assume that \( \delta_\psi S_\pm = 0 \). We consider a non-degenerate finite quadratic form
\[ \gamma_\pm = \begin{cases} 
0 & \text{if } \delta_\psi = 0 \text{ (equivalently, } v = 0), \\
q_1(2) \oplus q_{-1}(2) & \text{if } \delta_\psi = 1 \text{ and } c_v \equiv 0 \mod 4, \\
q_1(2)^2 & \text{if } c_v \equiv 2 \mod 4, \\
q_{\pm 1}(2) & \text{if } c_v \equiv \pm 1 \mod 4.
\end{cases} \]

Let \( v_{\gamma_\pm} \) be the characteristic element of \( \gamma_\pm \), which is unique. We have \( \gamma_\pm(v_{\gamma_\pm}) = c_v/2 \mod 2 \).

We denote \( (q_{S_\pm})_v := \langle v \oplus v_{\gamma_\pm}, (q_{S_\pm})_v \rangle \)
\[ = \frac{\det K((q_{S_\pm})_v)}{|A_{S_\pm}|^2} \mod \mathbb{Z}/2\mathbb{Z}. \]
There exists a unique even 2-adic lattice \( K((q_{S_\pm})_v) \) having the discriminant quadratic form \( (q_{S_\pm})_v \) and the same rank as the form \( (q_{S_\pm})_v \).

The invariant 
\[ \varepsilon_{v_{\pm}} \in \mathbb{Z}/2\mathbb{Z} \]
is defined by
\[ 5^{r_{\pm}} \equiv \frac{\det K((q_{S_\pm})_v)}{|A_{S_\pm}|^2} \mod \mathbb{Q}_2^2, \]
where \( \mathbb{Q}_p \) is the field of \( p \)-adic integers, \( A_q \) denotes the group where a finite form \( q \) is defined, and \( |A_q| \) denotes the order of \( A_q \). Thus we get the numerical invariants
\[ (\delta_\psi S_+, \delta_\psi S_-, \varepsilon_{v_+}, \varepsilon_{v_-}). \]

**Theorem 3.11** ([5]) **CONDITIONS 1.8.1** and **CONDITIONS 1.8.2** in [5] are sufficient and necessary for the existence of integral involutions of the lattice \( \mathbb{L}_K3 \) with condition \((S, \theta)\). (These conditions use the invariants (3.3) and the invariants \( r, \alpha, \delta_\psi S_+ \) and \( \delta_\psi S_- \)).

### 4 Applications to the classifications of real 2-elementary K3 surfaces

#### 4.1 Real 2-elementary K3 surfaces

Let \( X \) be a \( K3 \) surface with a non-symplectic holomorphic involution \( \tau \). We often call it 2-elementary K3 surfaces \((X, \tau)\) (see [4], [1], [6]). For a 2-elementary K3 surface \((X, \tau)\), we set
\[ H_{2_+}(X, \mathbb{Z}). \]
Definition 4.1 (Real 2-elementary K3 surfaces) We say that a triple \((X, \tau, \varphi)\) is a real K3 surface with non-symplectic holomorphic involution (or real 2-elementary K3 surface) if (1) \((X, \tau)\) is a K3 surface \(X\) with a non-symplectic holomorphic involution \(\tau\), (2) \(\varphi\) is an anti-holomorphic involution on \(X\), and (3) \(\varphi \circ \tau = \tau \circ \varphi\).

Let \(L_{K3}\) and \((S, \theta)\) be as in Subsection 3.2.

Definition 4.2 (Real 2-elementary K3 surfaces of type \((S, \theta)\)) Let \((X, \tau, \varphi)\) be a real 2-elementary K3 surface. We say that \((X, \tau, \varphi)\) is a real 2-elementary K3 surface (or real K3 surface with non-symplectic involution \(\tau\)) of type \((S, \theta)\) if there exists an isometry (so-called “marking”)

\[
\alpha : H_2(X, \mathbb{Z}) \cong L_{K3}
\]

such that \(\alpha(H_{2+}(X, \mathbb{Z})) = S\) and the following diagram commutes:

\[
\begin{array}{ccc}
H_{2+}(X, \mathbb{Z}) & \xrightarrow{\alpha} & S \\
\varphi \downarrow & & \downarrow \alpha \\
H_{2+}(X, \mathbb{Z}) & \xrightarrow{\alpha} & S.
\end{array}
\]

See [6], [7], [8] and [9] for more informations about real 2-elementary K3 surfaces.

4.2 Enumeration of integral involutions of the lattice \(L_{K3}\) with condition \(((3, 1, 1), -\text{id}_S)\)

Now let us consider real 2-elementary K3 surfaces \((X, \tau, \varphi)\) of type \((S, \theta)\) with

\[
(r(S), \alpha(S), \delta(S)) = (3, 1, 1) \text{ and } \theta = -\text{id}_S.
\]

Real 2-elementary K3 surfaces of this type are investigated in [7], [8] and [9]. As the first step for the geometric classification, Nikulin and Saito [7] enumerated all the isometry classes of integral involutions of \(L_{K3}\) of type \(((3, 1, 1), -\text{id}_S)\). In this subsection, we explain the details of this enumeration in [7].

We first remark that the lattice \(S \cong (3, 1, 1)\) is represented by the Gram matrix

\[
U \oplus (-2)
\]

with respect to some basis, let us say, \(\{\xi, \xi', \eta\}\).

Then the dual space \(S^* := \text{Hom}(S, \mathbb{Z})\) is generated by \(\xi^* = \xi', (\xi')^* = \xi\) and \(\eta^* = -\frac{1}{2} \eta\). Hence, the discriminant group \(A_S\) is as follows.

\[
A_S := S^*/S \cong (\frac{1}{2} \eta) \cong \mathbb{Z}/2\mathbb{Z}.
\]

Hence, we have the rank \(\alpha(S) = 1\), and

\[
\left(\frac{1}{2} \eta\right) \cdot \left(\frac{1}{2} \eta\right) = -\frac{2}{4} = -\frac{1}{2} \ (\not\in \mathbb{Z}).
\]

Hence, the discriminant quadratic form \(q_S\) is isomorphic to \(q_{-1}(2)\) and the parity \(\delta(S) = 1\).

Since \(\theta = -\text{id}_S\), we have \(S_+ = \{0\}\) and \(S_- = S\). Hence, obviously, \(\Delta(S_+, S)^{(-4)} = \emptyset\) and \(W^{(-4)}(S_+, S) = \{\text{id}_S\}\). We can also verify \(\Delta(S_-, S)^{(-4)} = \emptyset\). Hence, we have \(W^{(-4)}(S_-, S) = \{\text{id}_S\}\).

By Lemma 3.6, in our case we have

\[
G = \{\text{id}_S\}.
\]

Let us calculate the invariants defined in Subsection 3.2.

The subgroups \(H_\pm\) are defined by

\[
H_\pm := \delta^{-1}_\pm(0)/2S_\pm \quad (\subset S_\pm/2S_\pm),
\]
where for each \(x_\pm \in S_\pm\) we define
\[
\delta_{x_\pm} := \left\{ \begin{array}{ll}
0 & \text{if } x_\pm \cdot L_{x_\pm} \equiv 0 \pmod{2}, \\
1 & \text{otherwise,}
\end{array} \right.
\]
and we get the functions \(\delta_\pm : S_\pm \to \mathbb{Z}/2\mathbb{Z}\) defined by \(x_\pm \mapsto \delta_{x_\pm}\).

Obviously, we have \(H_+ = \{0\}\), and \(H_-\) is a subgroup of \(2(S_0^- \cap (\frac{1}{2} S^-))/2S_-\). Note that
\[
2(S_0^+ \cap (\frac{1}{2} S^-))/2S_- \cong (S_0^+ \cap (\frac{1}{2} S^-))/S_- = A_{S_-} = A_8 = \langle \frac{1}{2} \eta \rangle.
\]

We have \(H := (H_+ \oplus H_-)/T = H_-\), and hence,
\[
H = \{0\} \text{ or } \langle \eta \rangle.
\]

The finite quadratic form
\[
q_\rho : H = H_- \to \mathbb{Z}/2\mathbb{Z}
\]
is defined by
\[
q_\rho = (-q_\rho_-)[H_-],
\]
where we regard the discriminant quadratic form \(q_{S_-}\) as the form on \(2S_0^+/2S_-\). Then \(q_\rho\) is just determined by the subgroup \(H = H_-\). Note that \(-q_{S_-} \cong q_1(2)\).

If \(\delta_{\psi S} = 0\), then we define \(v := s_\psi (\in H)\).

We get a list of genus invariants
\[
(r, a; H (= H_-); \delta_\psi, \delta_{\psi S}, v).
\]

(4.1)

By Proposition 3.9, two integral involutions with condition \((S, \theta)\) have the same genus if and only if the invariants (4.1) coincide.

(1) \(H = \langle \eta \rangle\) case. We have
\[
q_\rho(\eta) = -\left(\frac{1}{2} \eta \cdot \frac{1}{2} \eta\right) \mod 2 = -\left(\frac{-2}{4}\right) \mod 2 = -\left(\frac{-1}{2}\right) \mod 2 = \frac{1}{2} \in \langle \frac{1}{2} \eta \rangle/2\mathbb{Z}.
\]

Hence, we have
\[
q_\rho = q_1(2),
\]
and for the numerical invariants (recall Subsection 3.2), we have
\[
\delta_H = 1 \text{ (the parity of } q_\rho), \quad \sigma_\rho = 1 \text{ (the signature of } q_\rho), \quad \mu_\rho = 0, \quad k_\rho = 0.
\]

When \(\delta_{\psi S} = 0\), the element \(c_v \text{ (mod 4)}\) is defined by
\[
q_\rho(v) = \frac{1}{2} c_v \pmod{2}.
\]

We have
\[
c_v = \left\{ \begin{array}{ll}
0 & \text{if } v = 0, \\
1 & \text{if } v = [\eta].
\end{array} \right.
\]

(2) \(H = 0 \ (\Leftrightarrow a_H = 0)\) case. In this case the arguments are trivial.

In our case we have \(A_{S_-}^{(2)} = A_{S_-}\). Let \(\bar{v}\) be the (unique) characteristic element of the discriminant quadratic form \(q_{S_-}\). We have
\[
\bar{v} = \frac{1}{2} [\eta] \ (\in A_{S_-}).
\]

Then, we have
\[
\delta_{\psi S_-} = 0 \text{ if and only if } \delta_{\psi S} = 0 \text{ and } v = [\eta].
\]
Hence, we have
\[
\begin{aligned}
\delta_{\psi} S_{-} &= 1 \quad \text{if } H = 0, \\
\delta_{\psi} S_{-} &= 1 \quad \text{if } H = \langle \eta \rangle \text{ and } v = 0, \\
\delta_{\psi} S_{-} &= \delta_{\psi} S \quad \text{if } H = \langle \eta \rangle \text{ and } v = \eta.
\end{aligned}
\]

On the other hand, we have
\[
\delta_{\psi} S_{+} = \delta_{\psi}.
\]

Assume that \(\delta_{\psi} S_{\pm} = 0\). Recall the non-degenerate finite quadratic form \(\gamma_{\pm}\) and the characteristic element \(v_{\gamma_{\pm}}\) of \(\gamma_{\pm}\).

If \(\delta_{\psi} S_{-} = 0\), then \(\delta_{\psi} S = 0\), \(v = \eta\), \(c_{v} = 1\), and hence, \(\gamma_{-} = q_{1}(2)\) and \(v_{\gamma_{-}} = \eta\).

We set
\[
(q_{S_{-}})_{v} := (v \oplus v_{\gamma_{-}})_{q_{S_{-}} \oplus (v \oplus v_{\gamma_{-}})} / [v \oplus v_{\gamma_{-}}].
\]

If \(\delta_{\psi} S_{-} = 0\), then \((q_{S_{-}})_{v} = (v \oplus v_{\gamma_{-}})_{q_{S_{-}} \oplus (v \oplus v_{\gamma_{-}})} / [v \oplus v_{\gamma_{-}}] = ([v \oplus v_{\gamma_{-}}]_{q_{S_{-}} \oplus (v \oplus v_{\gamma_{-}})} / [v \oplus v_{\gamma_{-}}] = [\eta \oplus (v \oplus v_{\gamma_{-}})]_{q_{S_{-}} \oplus (v \oplus v_{\gamma_{-}})} / [v \oplus v_{\gamma_{-}}].
\]

Considering the bilinear form \(q_{-}(2) \oplus q_{1}(2)\), we get \((v \oplus v_{\gamma_{-}})_{q_{S_{-}} \oplus (v \oplus v_{\gamma_{-}})} / [v \oplus v_{\gamma_{-}}] = ([\eta \oplus (v \oplus v_{\gamma_{-}})]_{q_{S_{-}} \oplus (v \oplus v_{\gamma_{-}})} / [v \oplus v_{\gamma_{-}}].
\]

Hence, we have \((q_{S_{-}})_{v} = 0\).

There exists a unique even 2-adic lattice \(K((q_{S_{-}})_{v})\) having the discriminant quadratic form \((q_{S_{-}})_{v}\) and the same rank as the form \((q_{S_{-}})_{v}\).

The invariant \(\varepsilon_{v_{-}} (\in \mathbb{Z}/2\mathbb{Z})\) is defined as in Subsection 3.2. Since \((q_{S_{-}})_{v} = 0\), we have \(\varepsilon_{v_{-}} = 0\).

On the other hand, if \(\delta_{\psi} S_{+} = \delta_{\psi} = 0\), then \(\gamma_{+} = 0\), \((\delta_{\psi} S_{-} = 1)\) and \(v = v_{\gamma_{+}} = 0\). Hence, we have
\[
(q_{S_{+}})_{v} := (v \oplus v_{\gamma_{+}})_{q_{S_{+}} \oplus (v \oplus v_{\gamma_{+}})} / [v \oplus v_{\gamma_{+}}] = (0 \oplus 0)_{q_{S_{+}} \oplus (v \oplus v_{\gamma_{+}})} / [v \oplus v_{\gamma_{+}}] = 0,
\]

and hence, \(\varepsilon_{v_{+}} = 0\).

Now we obtain the following necessary and sufficient conditions for the existence of integral involutions of \(L_{K3}\) of type \((3, 1, 1), \langle v \rangle\) by Theorem 3.11.

**CONDITIONS 1.8.1 ([5]).**

**Type 0** \((\delta_{\psi} = 0)\):
\[
\begin{aligned}
\delta_{H} &= 0, \text{ (hence, } H = \{0\}). \\
a &\geq 0,
\end{aligned}
\]

1) \(a + r \equiv 0 \text{ mod } 2\); \quad 2) \(2 - r \equiv 0 \text{ mod } 4\).

If \(a = 0\) and \(\mu_{\rho} = 0\), then \(2 - r \equiv \sigma_{\rho} \text{ mod } 8\).

**Type Ia** \((\delta_{\psi} = 1 \text{ and } \delta_{\psi} S = 0)\):
\[
\begin{aligned}
1) \quad c_{v} &= a_{H} - k_{\rho} \text{ mod } 2, \\
2) \quad \text{If } \mu_{\rho} = 0, \text{ then } c_{v} \equiv \sigma_{\rho} \text{ mod } 4, \\
3) \quad a_{H} &= 1, \\
4) \quad \text{If } \delta_{H} = 0, \text{ then } k_{\rho} \geq 1, \\
5) \quad \text{If } \delta_{H} = 0 \text{ and } k_{\rho} = \mu_{\rho} = 1, \text{ then } c_{v} \equiv 2 \text{ mod } 4.
\end{aligned}
\]

\[
\begin{aligned}
a &\geq a_{H} + k_{\rho}, \\
1) \quad a + r \equiv 0 \text{ mod } 2, \\
2) \quad 2 - r \equiv c_{v} \text{ mod } 4,
\end{aligned}
\]

If \(a = a_{H} + k_{\rho}\) and \(\mu_{\rho} = 0\), then \(2 - r \equiv \sigma_{\rho} \text{ mod } 8\).

**Type Ib** \((\delta_{\psi} S = 1)\):
\[
\begin{aligned}
a &\geq a_{H} + k_{\rho} + 1, \\
a + r \equiv 0 \text{ mod } 2.
\end{aligned}
\]

1) \(a = a_{H} + k_{\rho} + 1 \text{ and } \mu_{\rho} = 0\), then \(2 - r \equiv \sigma_{\rho} \pm 1 \text{ mod } 8\).

2) \(a = a_{H} + k_{\rho} + 2 \text{ and } \mu_{\rho} = 0\), then \(2 - r \not\equiv \sigma_{\rho} + 4 \text{ mod } 8\).

**CONDITIONS 1.8.2 ([5]).**

1) \(1 \leq r \leq 18\);

2) \(r - a \geq 0\);

3) \(r + a \leq 2a_{H} + 18\).

1) \(\delta_{\psi} S_{+} = 0 \text{ and } r - a = 0\), then \(2 - r \equiv c_{v} \text{ mod } 8\).

2) \(\delta_{\psi} S_{-} = 0 \text{ and } r + a = 2a_{H} + 18\), then \(2 - r \equiv c_{v} \text{ mod } 8\).
Here we take $c_v \pmod{8}$ in $\{\pm 1, 0, 2\}$ (see [5], p.118).

As stated above, if $H = \langle [\eta] \rangle$, then we have $q_\rho = q_1(2)$, and get the numerical invariants $\delta_H = 1$, $\sigma_\rho = 1$, $\mu_\rho = 0$, $k_\rho = 0$.

If $\delta_{\psi,S} = 0$, then the element $c_v \pmod{4}$ is 0 (if $v = 0$) or 1 (if $v = [\eta]$). By CONDITIONS 1.8.1, we have $c_v \equiv 1 \pmod{2}$, and hence, $v = [\eta]$ and $c_v \equiv 1 \pmod{4}$.

If $H = \{0\}$ and $\delta_{\psi,S} = 0$, then $v = 0$, equivalently, $\delta_{\psi} = 0$. By CONDITIONS 1.8.1, $\delta_{\psi} = 0$ implies $H = \{0\}$. Hence, if $H = \langle [\eta] \rangle$ and $\delta_{\psi,S} = 0$, then $\delta_{\psi} = 1$, equivalently, $v = [\eta]$.

Thus the data (4.1) (cf. the data (3.2)) is equivalent to

\[
(H, r, a, \delta_{\psi,S}).
\] (4.2)

Finally, we obtain all the genera of integral involutions $\psi$ of type $((3,1,1),-\text{id}_S)$. For all the possible genera (4.2), see the tables in [7] or [8].

Moreover, in our case, we can see that each genus determines the unique isometry class (see [7]).

Thus we obtain all the isometry classes of integral involutions of type $((3,1,1),-\text{id}_S)$. There are 102 isometry classes:

12 classes of Type 0 ($\Rightarrow H = \{0\}$ by CONDITIONS 1.8.1),
12 classes of Type Ia,
39 classes of Type Ib with $H = \{0\}$, and
39 classes of Type Ib with $H = \langle [\eta] \rangle$.

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